# On double one-parametric optimization problems and some applications

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**Abstract**

This paper deals with a special class of quadratic optimization problems with quadratic constraints, which we call double parametric. We try to solve these problems using Pathfollowing Methods with a special case of the Standard Embedding. The singularity theory developed by Jongen, Jonker and Twilt plays a great role in our investigations. In cases where a jump from one connected component to another one in the set of local minimizers and in the set of generalized critical points is not possible, we prove that the turning points are in negative direction. We survey results with respect to the choice of the starting point and make some proposals in order to overcome cases where jumps are not possible. We show the role of the Mangasarian-Fromovitz constraints qualification in the existence of the jumps and the solution of considered problems. Some examples of applications to global quadratic optimization and multicriteria optimization are given.

**Keywords**: parametric optimization, pathfollowing methods, jumps, generalized critical point, turning point in the negative sense

# 1 Introduction

In this paper, we consider the following double quadratic optimization problems

$$(\mathbf{P}_i) \qquad \min\{(x-x^o)^T D(x-x^o) \mid x \in M_i\} \qquad i=1,2.$$

$$M_1 \quad := \quad \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{lcll} g_j(x) & = & a_j^T x + b_j & \leq \; 0, \quad j \in J \\ g_{s+i}(x) & = & x^T A_i x + a_{s+i}^T x + b_{s+i} & \leq \; 0, \quad i \in I \end{array} \right\}$$

$$M_2 \quad := \quad \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{lcll} g_j(x) & = & a_j^T x + b_j & \leq \; 0, \quad j \in J \\ g_{s+1}(x) & = & x^T A_1 x + a_{s+1}^T x + b_{s+1} & \leq \; 0 \end{array} \right\}$$

$$J \quad = \quad \{1, \cdots, s\} \quad I = \{1,2,3\},$$

where $D$ is a positive-definite matrix, $A_1$ a negative semi-definite matrix, $A_2$ and $A_3$ positive semi-definite matrices, $a_j$ a vector in $\mathbb{R}^n$ and $x^o \in \mathbb{R}^n$ a specially chosen point. According to the properties of the matrices $A_i$,$i = 1, \ldots, 3$, the functions $g_{s+i}$ are the respective quadratic concave and convex constraints (for $i = 2, 3$).

Let

$$\begin{aligned} \mathcal{P} &:= \{x \in \mathbb{R}^n \mid a_j^T x + b_j \leq 0, \quad (j = 1, \cdots, s)\}, \\ G_1 &:= \left\{x \in \mathbb{R}^n \mid g_{s+1}(x) = x^T A_1 x + a_{s+1}^T x + b_{s+1} \leq 0\right\}, \\ G_2 &:= \{x \in \mathbb{R}^n \mid g_{s+2}(x) = x^T A_2 x + a_{s+2}^T x + b_{s+2} \leq 0\}, \\ G_3 &:= \{x \in \mathbb{R}^n \mid g_{s+3}(x) = x^T A_2 x + a_{s+3}^T x + b_{s+3} \leq 0\}. \end{aligned}$$

The feasible sets $M_1$ and $M_2$ can also be given by

$$\begin{aligned} M_1 &= \mathcal{P} \cap \bigcap_{i=1}^{3} G_i \\ M_2 &= \mathcal{P} \cap G_1. \end{aligned}$$

We notice that the sets $M_1$ and $M_2$ are not necessarily convex (see Fig. 1).

**Remark 1.** *The kind of problems $P_i$, $i = 1, 2$, was considered e.g. in [13].*

Let us introduce some conditions, that we shall need later. First we assume:

**(C1)** $\mathcal{P}$ is a convex polyhedron;

Then the sets $M_1$ and $M_2$ are compact. The sets

$$\begin{aligned} H_1 &= \{x \in \mathbb{R}^n \mid D_x g_{s+3}(x) = 2A_3 x + a_{s+3} = 0\} \\ H_2 &= \{x \in \mathbb{R}^n \mid D_x g_{s+1}(x) = 2A_1 x + a_{s+1} = 0\} \end{aligned}$$

represent the affine subspaces with the dimension
$n - rank(A_3)$ and $n - \mathrm{rank}(A_1)$, respectively.

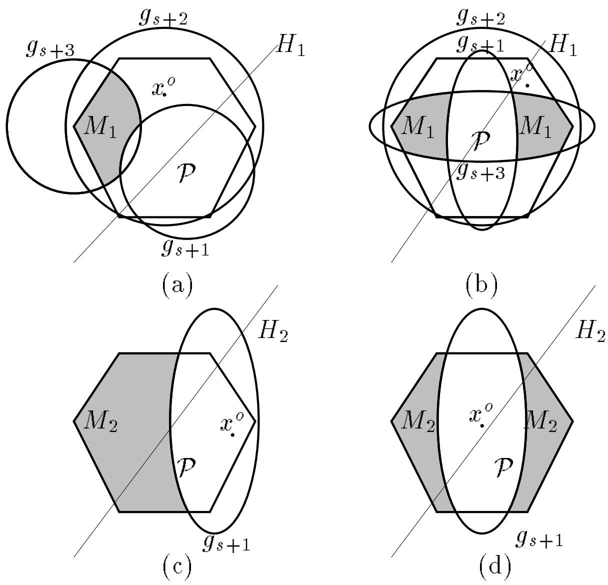



Figure 1: possible structure of $M_1$ and $M_2$

## Starting situation

One of the important issues we have to deal with is the choice of the starting point $x^o$. We shall see later that solving the problems $(P_i)$ $i = 1, 2$ successfully will depend on the choice of this starting point. In this paper we suggest to choose the starting point in the following way (see Fig. 1):

Problem $P_1$ :

**C2(a)** $\quad x^o \in int(\mathcal{P} \cap G_1 \cap G_2) \setminus (H_1 \cup G_3)$.

Problem $P_2$ :

**C2(b)** $\quad x^o \in int\mathcal{P} \setminus (G_1 \cup H_2)$. Furthermore, we claim that the observed constraints must have full dimension(cf. [13]):

**(C3)** $$M_i = \text{cl int} M_i \qquad i = 1, 2.$$

With the condition $(C3)$ we want to exclude cases where

$$M_1 \subset \partial(\mathcal{P} \cap \bigcap_{i=1}^{3} G_i), \qquad M_2 \subset \partial(\mathcal{P} \cap G_1).$$

Let us recall now the well-known concept of embedding (cf. e.g. Allgower [1], Guddat et al. [5], Dentcheva [3], Gfrerer et al. [10]) and propose the following embeddings in order to solve $(P_i), i = 1, 2$:
Let $x^o \in \mathcal{P}$ be arbitrarily chosen and define

**$\mathbf{P_1}(t)$ :** $$\min\{f(x,t) \mid x \in M_1(t)\} \quad t \in (-\infty, 1],$$

where

$$\begin{aligned} f(x,t) &:= (x-x^o)^T D(x-x^o) \\ M_1(t) &:= \{x \in \mathbb{R}^n \mid g_j(x,t) \leq 0, \quad (j=1,\cdots,s+3)\} \\ g_j(x,t) &:= g_j(x), \quad j=1,\cdots,s+2 \\ g_{s+3}(x,t) &:= g_{s+3}(x)+(t-1)g_{s+3}(x^o) \end{aligned}$$

and

$\mathbf{P_2}(t)$ : $$\min\{f(x,t) \mid x \in M_2(t)\} \quad t \in (-\infty, 1],$$

where

$$\begin{aligned} f(x,t) &:= (x-x^o)^T D(x-x^o) \\ M_2(t) &:= \{x \in \mathbb{R}^n \mid g_j(x,t) \leq 0, \quad (j=1,\cdots,s+1)\} \\ g_j(x,t) &:= g_j(x), \quad j=1,\cdots,s \\ g_{s+1}(x,t) &:= g_{s+1}(x)+(t-1)g_{s+1}(x^o) \end{aligned}$$

In Section 2 we present the necessary background for one-parametric problems, of which $P_i(t)$ is a particular case. In Section 3 we study $P_i(t)$ and in section 4 we analyze under which perturbations on $P_i(t)$ we obtain a parametric problem $P_{\mathcal{D}}(t)$ belonging to the generic class $\mathcal{F}$ of Jongen, Jonker and Twilt [15]. In Section 5 we present some applications.

## 2 Theoretical Background

We consider the general one-parametric problem:

$P(t)$ : $$\min\{f(x,t) \mid x \in M(t)\}, \quad t \in \mathbb{R},$$

where

$$\begin{aligned} M(t) &= \{x \in \mathbb{R}^n \mid h_i(x,t)=0, i \in I, \quad g_j(x,t) \leq 0, j \in J\}, \\ I &= \{1,\cdots,m\}, \quad m<n, \quad J=\{1,\cdots,s\} \end{aligned}$$

and $f, h_i, g_j \in C^q(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}), q \geq 2$.
Furthermore, we introduce the following notations:

$$\begin{aligned} \textstyle\sum_{gc} &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \; x \text{ is a g.c. point of } P(t)\}, \\ \textstyle\sum_{stat} &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \; x \text{ is a stationary point of } P(t)\}, \\ \textstyle\sum_{loc} &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \; x \text{ is a local minimizer of } P(t)\}, \\ H &:= (h_1,\cdots,h_m)^T, \qquad G := (g_1,\cdots,g_s)^T. \end{aligned}$$

**Note 1.** *For the definition of a g.c. point we refer to [15], see also [5].*

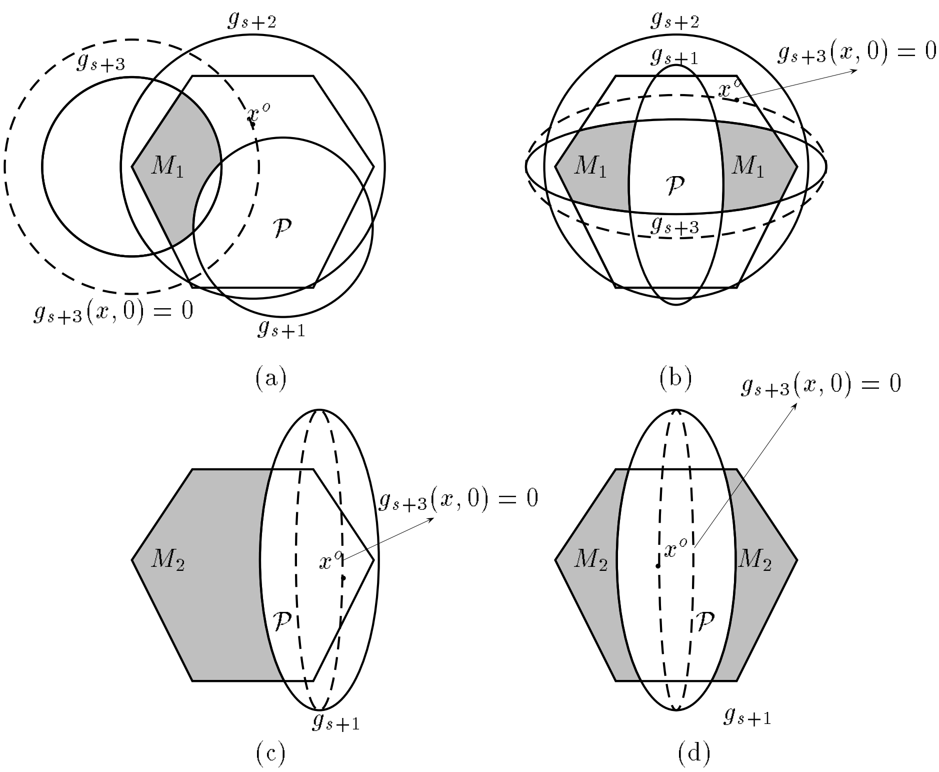



Figure 2: possible structure of $M_1(t)$ and $M_2(t)$

The Linear *Independence Constraint Qualification (LICQ)* is said to hold at $\bar{x} \in M(\bar{t})$ if the vectors

$$D_x h_i(\bar{x},\bar{t}), i \in I, Dg_j(\bar{x},\bar{t}), j \in J_o(\bar{x},\bar{t}),$$

are linearly independent $(J_o(x,t) := \{j \in J \mid g_j(x,t) = 0\})$.

The *Mangasarian-Fromovitz Constraint Qualification (MFCQ)* is satisfied at $\bar{x} \in M(\bar{t})$ if:

**MF1** $\{Dh_i(\bar{x},\bar{t}), i \in I\}$ are linearly independent,

**MF2** there exists a vector $\xi \in \mathbb{R}^n$ satisfying:

$$\begin{array}{rcll} Dh_i(\bar{x},\bar{t})\xi & = & 0, & i \in I \\ Dg_j(\bar{x},\bar{t})\xi & < & 0, & j \in J_o(\bar{x},\bar{t}). \end{array}$$

Next we cite our short characterization of the class $\mathcal{F}$ introduced by Jongen, Jonker and Twilt [15] from 2.5 in [5] . In [15] the local structure of $\sum_{gc}$ is completely described if $(f,H,G)$ belongs to a $C^3_s$-open and dense subset $\mathcal{F}$ of $C^3(\mathbb{R}^{n+1},\mathbb{R})^{1+m+s}$, where $C^3_s$ denotes the strong (or Whitney-)$C^3_s$-topology (see also [5]). A typical baseneighbourhood of this topology with $f \in C^k(\mathbb{R}^{n+1}, \mathbb{R})$ and $\phi \in C_+(\mathbb{R}^{n+1}, \mathbb{R})$ is given by

$$B_\phi^k(f) := \left\{ g \in C^k(\mathbb{R}^{n+1}, \mathbb{R}) \middle| \begin{array}{c} |\partial^\alpha g(x) - \partial^\alpha f(x)| < \phi(x) \\ \forall x \in \mathbb{R}^{n+1}, \forall \alpha \text{ with } |\alpha| \le k \end{array} \right\},$$

where

$$C_+(\mathbb{R}^{n+1}, \mathbb{R}) := \{\phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \mid \phi \text{ is continuous and } \phi(x) > 0 \quad \forall x \in \mathbb{R}^{n+1}\}.$$

If $(f, H, G) \in \mathcal{F}$ , then $\sum_{gc}$ can be divided into 5 types.

**Type 1:** A point $\bar{z} = (\bar{x}, \bar{t}) \in \sum_{gc}$ is of Type 1 if the following conditions are satisfied:
There exists $\bar{\lambda}_i, \bar{\mu}_j \in \mathbb{R}, i \in I, j \in J_o(\bar{z})$ satisfying

$$(1) \qquad \left( D_x f + \sum_{i \in I} \bar{\lambda}_i D_x h_i + \sum_{j \in J_o(\bar{z})} \bar{\mu}_j D_x g_j \right)_{|z=\bar{z}} = 0,$$

(**ND1**) LICQ is satisfied at $\bar{x} \in M(\bar{t})$,
therefore $\bar{\lambda}_i, \bar{\mu}_j \in \mathbb{R}, i \in I, j \in J_o(\bar{z})$ are uniquely defined,
(**ND2**) $\bar{\mu}_j \neq 0, j \in J_o(\bar{z})$,
(**ND3**) $D_x^2 L(\bar{x}, \bar{t}) \mid_{T(\bar{z})}$ is nonsingular, where $D_x^2 L$ is the Hessian of the Lagrangian

$$L(x,t) = f(x,t) + \sum_{i \in I} \bar{\lambda} h_i(x,t) + \sum_{j \in J_o(\bar{z})} \bar{\mu} g_j(x,t)$$

and the uniquely determined numbers $\bar{\lambda}_i, \bar{\mu}_j$ are taken from (1).
Furthermore,

$$T(z) = \{\xi \in \mathbb{R}^n \mid D_x h_i(z)\xi = 0, D_x g_j(z)\xi = 0, j \in J_o(z)\}$$

is the tangent space at $z$. $D_x^2 L(z) \mid_{T(z)}$ represents $V^T D_x^2 L V$, where $V$ is a matrix whose columns form a basis of $T(z)$.
A point of Type 1 is a nondegenerate critical point. The set $\sum_{gc}$ is the closure of the set of all points of Type 1, the points of the Types $2-5$ constitute a discrete subset of $\sum_{gc}$.
The points of the Types $2-5$ represent basic degeneracies:

**Type 2:** - violation of $(ND2)$

**Type 3:** - violation of $(ND3)$

**Type 4:** - violation of $(ND1)$ and $|I| + |J_o(\bar{z})| - 1 < n$

**Type 5:** - violation of $(ND1)$ and $|I| + |J_o(\bar{z})| = n + 1$

For each of these five types Figure 3 illustrates the local structure of $\sum_{gc}$ in the neighbourhood of stationary points. The full curves stand for the curve of stationary points $z = (x, t)$, and the dotted curve represents the curve of g.c. points which are not stationary points. Let $\sum_{gc}^{\nu}, \nu \in \{1, \cdots, 5\}$ be the set of g.c. points of Type $\nu$. The class $\mathcal{F}$ is defined by

$$\mathcal{F} := \left\{ (f, H, G) \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s} \mid \sum_{gc} \subset \bigcup_{\nu=1}^{5} \sum_{gc}^{\nu} \right\}.$$

Recall that $\mathcal{F}$ is $C_s^3$ open and dense in $C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s}$(cf. [15]).

## The algorithm PATH III

This algorithm computes a numerical description of a compact connected component in $\sum_{gc}$, i.e., in particular, it finds a discretization of the interval $[t_A, t_B], t_A < 0 < t_B$(not necessarily $[t_A, t_B] \supset [0, 1]$) and corresponding g.c. points starting at $(x^o, 0) \in \sum_{gc}$. The algorithm is based on the active index set strategy and is a so-called predictor-corrector scheme if the active index set is constant. A Newton corrector is used.

The main point of the approach consists in the computation of the new index sets for the possible continuations.

We note that we do not have any numerical difficulties walking around turning points of the Types 3 or 4.

More precisely: if there exists a $PC^2$-path connecting $(x^o, 0)$ and a point $(x^*, 1)$, then we obtain, in a finite number of predictor and corrector steps, a point lying in the radius of convergence of the Newton method for $x^*$ with respect to the problem $P(1)$. Since PATH III is not successful in finding a point $(x^*, 1) \in \sum_{gc}$ in general, we propose to jump from one connected component in $cl \sum_{loc}$ and $\sum_{gc}$, respectively, to another one. For more information we refer the reader to Guddat et al. [5] and PAFO [11].

## The algorithm JUMP I

This algorithm works in the set $cl \sum_{loc}$. Starting at the known local minimizer $x^o$ at $t_o = 0$, a connected component in $cl \sum_{loc}$ for increasing $t$ will be numerically described by using PATH III. Depending on the appearance of a singularity, a direction of descent will be computed. Using a feasible direction method, a local minimizer on another connected component in $\sum_{loc}$ will be calculated and PATH III starts again. We have to take into account that we have no proposals for jumps in any case. Jumps are possible if there occurs a turning point of Type 2 or a point of Type 3. Let $t$ be near $\bar{t}, t < \bar{t}$, and let $x_m(t)$ and $x_s(t)$ be the local minimizer and a point of $\sum_{stat} \setminus \sum_{loc}$, respectively. Then, as $t$ tends to $\bar{t}$, the vector

$$u(t) := \frac{x_s(t) - x_m(t)}{||x_s(t) - x_m(t)||}$$

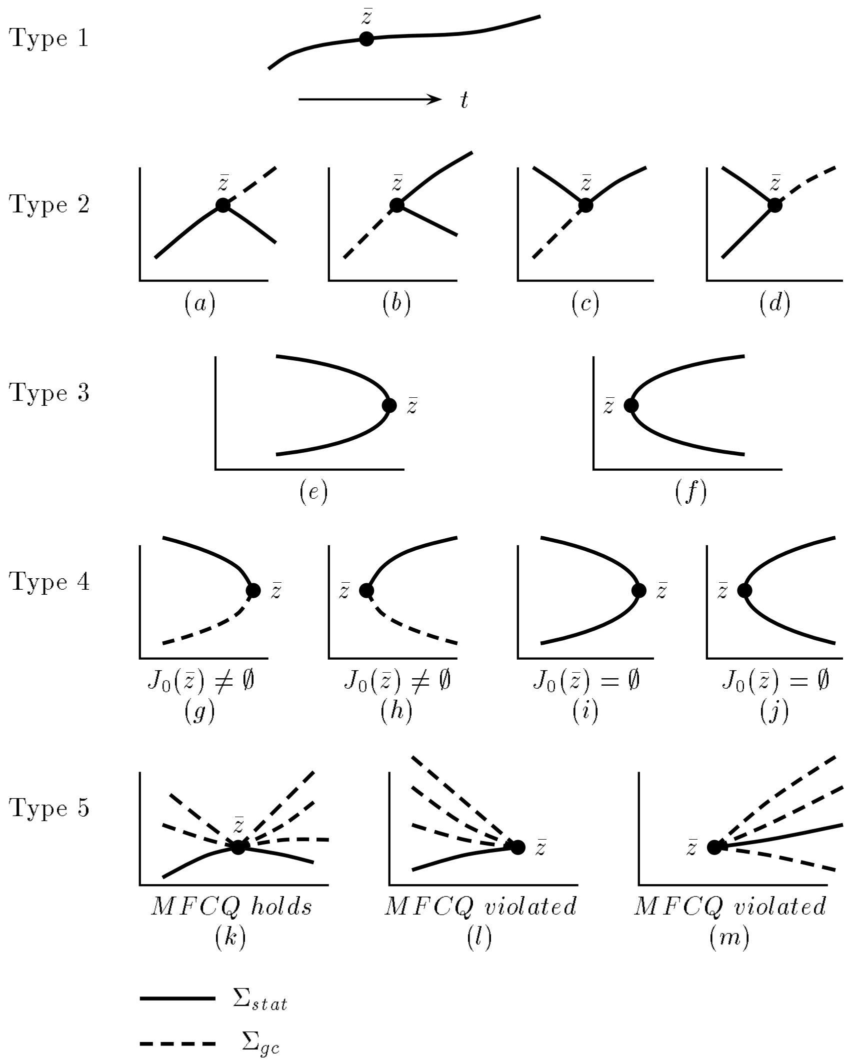



Figure 3: 5 Types of Jongen, Jonker and Twilt

tends to a tangential vector, say $\bar{u}$, which is a direction of (higher order) descent. Hence, for $t$ near $\bar{t}, t < \bar{t}$, the vector $x_s(t) - x_m(t)$ provides an approximately tangential direction of descent (see Figure 4)

A g.c. point of Type 4 is a quadratic turning point and, when passing $\bar{z}$ along $cl \sum_{loc}$, the local minimizer switches into a local maximizer. We have the following cases for $t < \bar{t}$ and $t$ close to $\bar{t}$:

**Case** $I$**:** The value of $f$ decreases.

**Case** $II$**:** The value of $f$ increases.

In Case $I$ it is possible to jump to another brach of local minimizers. In fact, since the feasible set $M(t)$ is compact, we compute a point on $\sum_{gc}$ beyond the turning point, say $(x_{max}(t), t)$ with $t < \bar{t}$ close to $\bar{t}$. The point $x_{max}(t)$ is a local maximizer for $P(t)$ and we can start at $x_{max}(t)$ with a descent method in order to find a local minimizer. In Case $II$, there is no proposal for possible jumps (see Fig. 4).

If a point of Type 5 appears, we also do not know a jump in case the MFCQ is violated. Such a situation is characterized by the fact that the connected component in the feasible set shrinks to one point and becomes empty for increasing $t$.

## The algorithm JUMP II

The algorith Jump II works in $\sum_{gc}$ and it is useful for the computation of as many connected components as possible using PATH III and jumps in $\sum_{gc}$. For more details, we refer the reader to Guddat et al. [5] for the pathfollowing algorithms in the Sections 4 and 5.

# 3 Properties of the double parametric embedding

The first theorem includes basic properties of the embeddings $P_i(t)$, $i = 1, 2$.

**Theorem 1.** *It holds:*

$(E1)$ *$x^o$ is a global minimizer for $P_i(0)$.*

$(E2)$ *$P_i(1) \equiv P_i$,*

$(E3)$ *$M_i(t_1) \supseteq M_i(t_2)$ for $t_1 \leq t_2$, $t_1, t_2 \in (-\infty, 1]$,*

$(E4)$ *$M_i(t) \neq \emptyset$ and compact , for all $t \in (-\infty, 1]$,*

$(E5)$ *$\psi_i(t) \neq \emptyset \quad \forall t \in (-\infty, 1]$, where $\psi_i(t)$ denotes the set of all global minimizers of $P_i(t)$.*

*Proof.* Without loss of generality we consider the problem $P_1(t)$. $(E1)$ and $(E2)$ are obvious.
$(E3)$ Let $t_1 \leq t_2$, $t_1, t_2 \in (-\infty, 1]$. If $x \in M_1(t_2)$, then $g_{s+3}(x) + (t_2 - 1)g_{s+3}(x^o) \leq 0$.
We have $g_{s+3}(x) + (t_1 - 1)g_{s+3}(x^o) \leq g_{s+3}(x) + (t_2 - 1)g_{s+3}(x^o)$,
since $g_{s+3}(x^o) > 0$. So, $x \in M_1(t_1)$.
$(E4)$ follows from (E3) and $M_i \neq \emptyset$. $M_1(t)$ is compact, since $\mathcal{P}$ is compact and the set $\bigcap_{i=1}^{3} G_i$ closed.
$(E5)$ follows from $(E4)$. $\square$

For the problems $P_i(t)$, $i = 1, 2$ let us assume:
**(C4)** $(f, g_1, \cdots, g_{s+3}) \in \mathcal{F}$ and $(f, g_1, \cdots, g_{s+1}) \in \mathcal{F}$, respectively .
$\mathcal{F}$ is the class of Jongen, Jonker and Twilt.

**Remark 2.** *In this case the problems $P_i(t)$, $i = 1, 2$ are regular in the sense of Jongen, Jonker and Twilt (i.e., JJT-regular).*

One important issue is to determine the kind of singularities which may appear. Theorem 2 gives us an answer.

**Definition 1.** *A point $(\bar{x}, \bar{t}) \in M_i(t) \times \mathbb{R}, i = 1, 2$ is a vertex if $\bar{x}$ is a vertex of the polyhedron $\mathcal{P}$.*

**Theorem 2 (Singularity Theorem).** *Assume that $(C1)$, $(C2)$, $(C3)$, $(C4)$ hold and let*

(2)
$$I_o^k(\bar{x}, \bar{t}) := \{j_i \in \{1, \cdots, s\} \mid g_{j_i}(x, t) = a_{j_i}^T \bar{x} + b_{j_i} = 0, \quad i = 1, \cdots, k\},$$

*be the index-set of active linear inequalities of the problems $P_i(t)$, $i \in \{1, 2\}$. Then the following assertions are true:*

**A)** *For* $\mathbf{P(t) := P_1(t)}$
*For any $\bar{z} = (\bar{x}, \bar{t}) \in cl \sum_{loc} \setminus \sum_{gc}^1$ and $\bar{t} \in (-\infty, 1]$, the following holds:*

- *(i)* $\bar{z} \in \sum_{gc}^2 \cup \sum_{gc}^3 \cup \sum_{gc}^4 \cup \sum_{gc}^5$.
- *(ii)* $(x^o, 0) \in \sum_{gc}^2 \cap \sum_{loc}$ *and* $(x^o, 0)$ *is no turning point.*
- *(iii) Each regular vertex $\bar{z}$ with $J_o(\bar{z}) = \{s + 3\} \cup I_o^k(\bar{z})$, is a point of Type 5 and the MFCQ is fulfilled.*
- *(iv) If $\bar{z}$ is no vertex of $\mathcal{P}$ and $J_o(\bar{z}) = \{s + 3\} \cup I_o^k(\bar{z})$, or $J_o(\bar{z}) = \{s + 2\} \cup \{s + 3\} \cup I_o^k(\bar{x}, \bar{t})$, then*
  *$\bar{z} \in \sum_{gc}^2 \cup \sum_{gc}^3 \cup \sum_{gc}^5$ and the MFCQ is fulfilled.*
- *(v) If $s + 1 \in J_o(\bar{z})$, then it holds:*
  - *(a) If the MFCQ is not satisfied at $\bar{z} \Longrightarrow \bar{z} \in \sum_{gc}^4 \cup \sum_{gc}^5$,*
  - *(b) If the MFCQ is fulfilled at $\bar{z} \Longrightarrow \bar{z} \in \sum_{gc}^2 \cup \sum_{gc}^3 \cup \sum_{gc}^5$.*

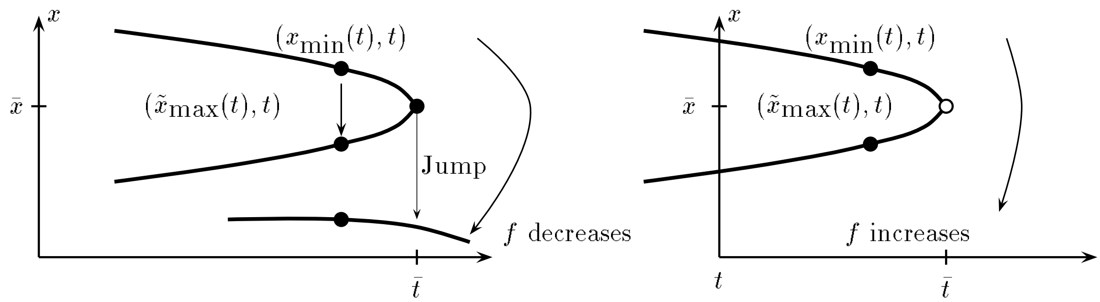



Figure 4: Jump in Points of Type 4

**B**) *For* $\mathbf{P(t)} := \mathbf{P_2(t)}$
*For any* $\bar{z} = (\bar{x}, \bar{t}) \in cl \sum_{loc} \setminus \sum_{gc}^1$ *and* $\bar{t} \in (-\infty, 1]$, *the following holds:*

*(i)* $\bar{z} \in \sum_{gc}^2 \cup \sum_{gc}^3 \cup \sum_{gc}^4 \cup \sum_{gc}^5$.

*(ii)* $(x^o, 0) \in \sum_{gc}^2 \cap \sum_{loc}$ *and* $(x^o, 0)$ *is no turning point.*

*(iii) If* $(\bar{x}, \bar{t})$ *is a regular vertex with* $J_o(\bar{z}) = \{s+1\} \cup I_o^k(\bar{z})$, *then* $\bar{z} \in \sum_{gc}^5$.

*(iv) If* $\bar{z}$ *is no vertex of with* $J_o(\bar{z}) = \{s+1\} \cup I_o^k(\bar{x}, \bar{t})$, *then it holds:*

*(a) If the MFCQ is violated* $\Longrightarrow$ $\bar{z} \in \sum_{gc}^4 \cup \sum_{gc}^5$,

*(b) If the MFCQ is satisfied* $\Longrightarrow$ $\bar{z} \in \sum_{gc}^2 \cup \sum_{gc}^3 \cup \sum_{gc}^5$.

**C**) *If* $\bar{z} \in cl \sum_{loc} \cap \sum_{gc}^4$, *then*

$$sign\left(-D_t \sum_{j=1}^{p} \mu_j g_j(\bar{z})\right) < 0.$$

**Remark 3. C)** *means that the jump is impossible.*

*Proof.* **A**) $\mathbf{P(t)} := \mathbf{P_1(t)}$

($i$) Follows from $(ii) - (v)$.

($ii$) From the choice of $x^o$ in (**V2**) we have:

(a)

$$g_{s+3}(x^o, t) \begin{cases} < 0 & \text{if } t < 0 \\ = 0 & \text{if } t = 0 \\ > 0 & \text{if } t > 0, \end{cases}$$

(b) $g_j(x^o, t) = g_j(x^o) \neq 0, j \neq s+3$,

(c) it follows from $(a)$ and $(b)$ that $J_o(x^o, 0) = \{s+3\}$.

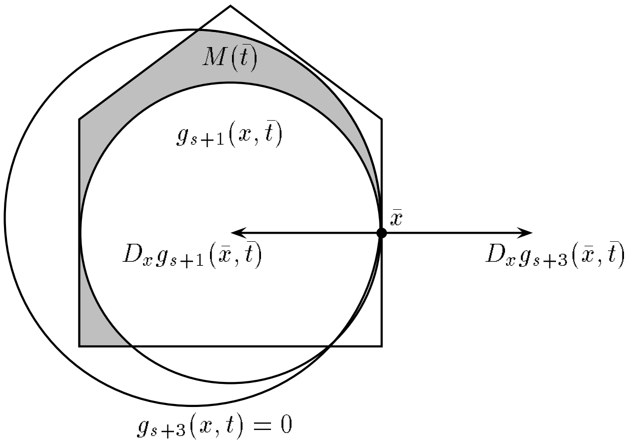



Figure 5: Violation of the MFCQ

Since $x^o$ is a global minimizer of $P_1(t)$, $t \leq 0$ we also have that $x^o$ is a g.c. point. Next, we verify that $D_x g_{s+3}(x^o) \neq 0$, since $x^o \notin H_1$, $|J_o(x^o,0)| = 1$, and $D_x^2 L(x^o,0) = D$. Consequently the LICQ is satisfied and the first conclusion follows from (C4). The latter statement follows from the fact that $D_x^2 L(x^o,0)/T(x^o,0)$ is positive-definite and the arguments described in (5.2.20) [5].

$(iii)$ Since $\bar{z}$ is a regular vertex, we have $|J_o(\bar{z})| = |I_o^k(\bar{z})| + 1 = n+1$. Consequently, we obtain the first statement. Using the convexity of $g_j, j \in J_o(\bar{x},\bar{t})$ we have

$$D_x g_j(\bar{x},\bar{t})((x,\bar{t}) - (\bar{x},\bar{t})) \leq g_j(x,\bar{t}).$$

With our choice of $\hat{x} \in \text{int } M_1 \subset M_1(\bar{t})$, we obtain $g_j(\hat{x},\bar{t}) < 0$. We set $\xi := (\hat{x},\bar{t}) - (\bar{x},\bar{t})$. Here $\xi$ is an MFCQ-vector. For a given $\xi$ we have $D_x g_j(\bar{x},\bar{t})\xi < 0$. Hence the MFCQ is satisfied.

$(iv)$ Readily the same as $(iii)$. Of course, if $|J_o(\bar{z})| = n+1$ ( $|I_o^k(\bar{z})| = n-1$) and $J_o(\bar{z}) = \{s+2\} \cup \{s+3\} \cup |I_o^k(\bar{z})|$, then $\bar{z}$ is a point of Type 5 and the $MFCQ$ is satisfied.

$(v)$ $(a)$ follows from Theorem 2.5.4 in [5] and $(b)$ is obvious.

**B**) $P(t) := P_2(t)$ For $(i)$, $(ii)$ and $(iv)$ we proceed as in **A)**
For $(iii)$ we also proceed as in **A)**. Here the MFCQ will not hold in general (see Fig. 5).

**C**) In view of (5.2.31) and (5.2.32) in [5], it suffices indeed to check that

$$sign\left(-D_t \sum_{j=1}^{p} \mu_j g_j(\bar{z})\right) < 0.$$

For this purpose we have

$$D_t\left(\sum_{j=1}^{p}\mu_j g_j(\bar{z})\right) = u_{\hat{j}} g_{\hat{j}}(x^o), \quad \hat{j} = s+1, s+3.$$

Since $\bar{z} \in \sum_{gc}^4$, the MFCQ is not satisfied, we have $\mu_{\hat{j}} > 0$. From $g_{\hat{j}}(x^o) > 0$ we obtain:

$$D_t\left(\sum_{j=1}^{p} u_j g_j(\bar{z})\right) > 0.$$

Hence $\delta = sign(-D_t\mathcal{L}(\bar{z})) < 0$. This completes the proof. $\square$

**Remark 4.** *Let $z = (x,t) \in \sum_{gc}$ be a degenerate vertex and let $s+3 \in J_o(z)$(for the Problem $P_1$) or $s+1 \in J_o(z)$( for the Problem $P_2$), then it holds that $(f,G) \notin \mathcal{F}$.*

From the practical point of view, we follow a path in $\sum_{gc}$, starting at a point of Type 1.

**Theorem 3.** *Assume that $(C4)$ for $P_i(t)$, $i = 1,2$. Then it holds:*

$$(x^o, t_o) \in \textstyle\sum_{gc}^1, \quad \forall t_o < 0.$$

*Proof.* Follows immediately from the fact that $g_{s+3}(x^o,t_o) < 0$, $D_x f(x^o,t_o) = 0$ and $D_x^2 f(x^o,t_o) = 2D$ for all $t_o < 0$. $\square$

## 3.1 Illustrative examples

**Example 3.1.**

$$\begin{aligned} \min\{f(x_1,x_2) &:= (x_1 - {x_1}^o)^2 + (x_2 - {x_2}^o)^2 | (x_1,x_2) \in M\} \\ M &:= \{(x_1,x_2) \in \mathbb{R}^2 | g_j(x_1,x_2) \le 0, j = 1,\cdots,7\} \end{aligned}$$

$$\begin{aligned} g_1(x_1,x_2) &:= x_1 - 6x_2 + 8 \\ g_2(x_1,x_2) &:= 3x_1 - x_2 - 27 \\ g_3(x_1,x_2) &:= x_1 + 3x_2 - 29 \\ g_4(x_1,x_2) &:= -6x_1 + x_2 + 22 \\ g_5(x_1,x_2) &:= -(x_1-7)^2 - (x_2-4.5)^2 + 9 \\ g_6(x_1,x_2) &:= (x_1-7)^2 + (x_2-5)^2 - 16 \\ g_7(x_1,x_2) &:= (x_1-3)^2 + (x_2-5.5)^2 - 9 \\ {x_1}^o = 10.5 & \qquad {x_2}^o = 5 \\ g_7(x_1,x_2,t) &:= g_7(x_1,x_2) + (t-1)g_7({x_1}^o,{x_2}^o) \end{aligned}$$

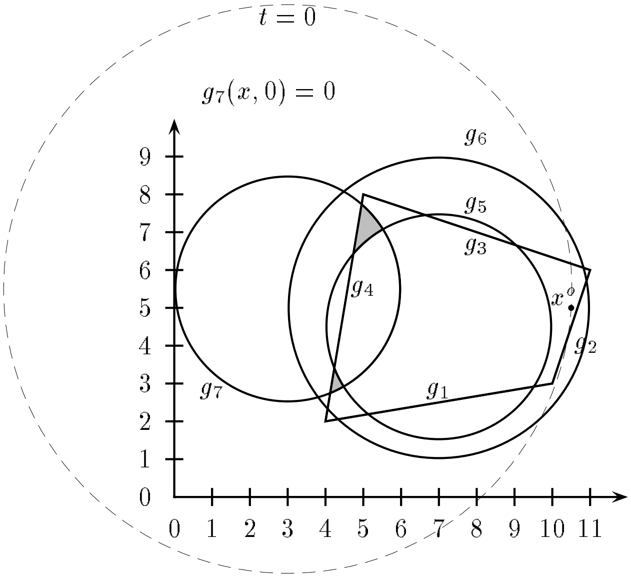



Figure 6: Feasible Set for the Example 3.1

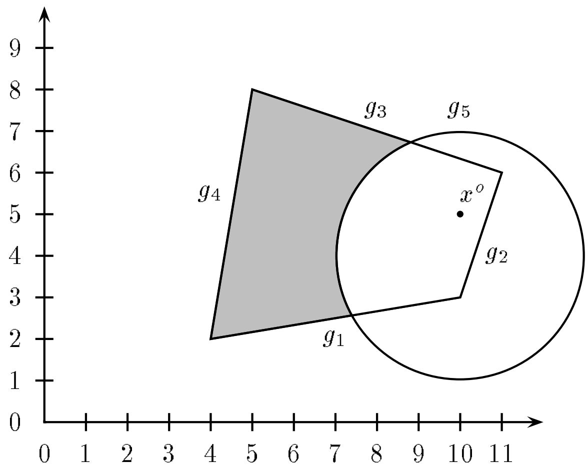



Figure 7: Feasible Set for the Example 3.2

**Example 3.2.**

$$\min\{f(x_1,x_2) := (x_1 - {x_1}^o)^2 + (x_2 - {x_2}^o)^2 | (x_1,x_2) \in M\}$$
$$M := \{(x_1,x_2) \in \mathbb{R}^2 | g_j(x_1,x_2) \leq 0, j = 1,\cdots,7\}$$

$$\begin{aligned}
g_1(x_1,x_2) &:= x_1 - 6x_2 + 8 \\
g_2(x_1,x_2) &:= 3x_1 - x_2 - 27 \\
g_3(x_1,x_2) &:= x_1 + 3x_2 - 29 \\
g_4(x_1,x_2) &:= -6x_1 + x_2 + 22 \\
g_5(x_1,x_2) &:= -(x_1 - 10)^2 - (x_2 - 4)^2 + 9 \\
{x_1}^o &= 10 \qquad {x_2}^o = 5 \\
g_5(x_1,x_2,t) &:= g_5(x_1,x_2) + (t-1)g_5({x_1}^o,{x_2}^o)
\end{aligned}$$

**Remark 5.** *The Figures 8, 9 and 10 show the dependence of the solution on the different starting points. If we start with the point $x^o = (10.5, 5)$, we will not be successful, but starting with $x^o = (9.55, 2.90)$ and $x^o = (4.5, 7.5)$ leads us to the solution. Figure 11 shows the solution of Example 3.2.*

## 3.2 On the role of the MFCQ

In this section we give an answer to the question *under which conditions the path-following methods are successful* with a specially chosen starting point, when we assume that $(C4)$ holds. Under the assumption that the MFCQ is satisfied at each $x \in M_i(t)$, points of Type 4 are excluded. Hence, the case $II$ in Section 2 may not appear. We ask for a condition on the original problems $(P_i), i = 1,2$. We discuss the so-called Enlarged Mangasarian-Fromovitz Constraint Qualification (briefly EnMFCQ) introduced by Gfrerer et al. [10].

**Definition 2 (EnMFCQ).** *The EnMFCQ is satisfied for $P_1$ if, for all $x \in \mathcal{P}$ (polyhedron), there exists a vector $\xi \in \mathbb{R}^n$ with*

$$\begin{aligned}
g_j(x) + Dg_j(x)\xi &< 0, \quad j \in J_+(x) := \{j \in J \mid g_j(x) \geq 0, j \neq s+2\} \\
D_x g_{s+2}(x)\xi &< 0, \quad \text{if} \quad g_{s+2}(x) = 0.
\end{aligned}$$

*The EnMFCQ is satisfied for $P_2$ if, for all $x \in \mathcal{P}$, there exists a vector $\xi \in \mathbb{R}^n$ with:*

$$g_j(x) + Dg_j(x)\xi < 0, \qquad j \in J_+(x) := \{j \in J | g_j(x) \geq 0\}.$$

The answer to the question above is given by the following theorem:

**Theorem 4.** *Assume the EnMFCQ is satisfied for $P_i, i = 1,2$. Then the MFCQ is satisfied for all $x \in M_i(t), i = 1,2$*

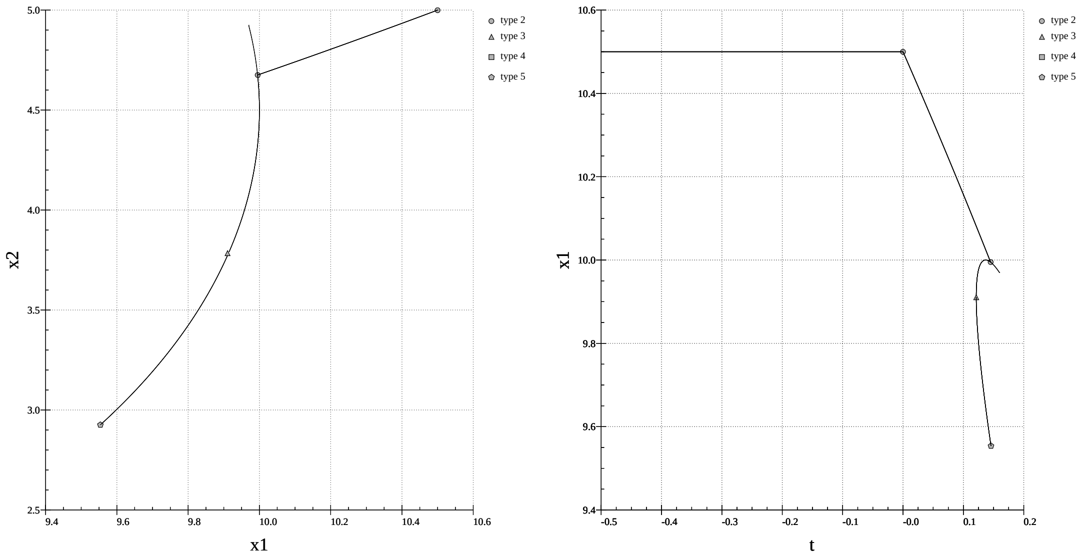



Figure 8: Example 1 in $(x_1, x_2)$ and $(t, x_1)$-space

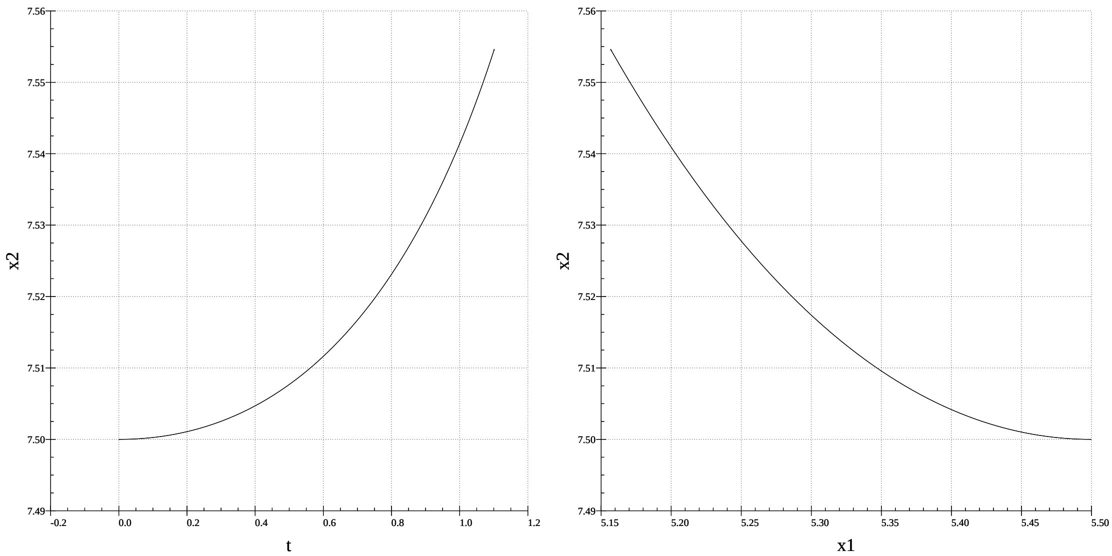



Figure 9: Example 1 in $(x_1, x_2)$ and $(t, x_1)$-space

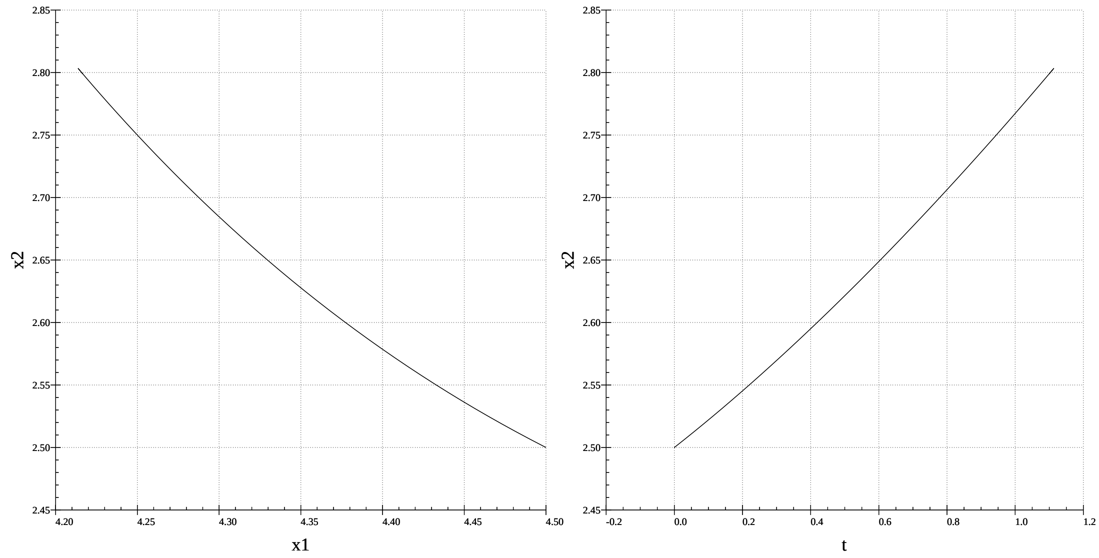



Figure 10: Example 1 in $(x_1, x_2)$ and $(t, x_1)$-space

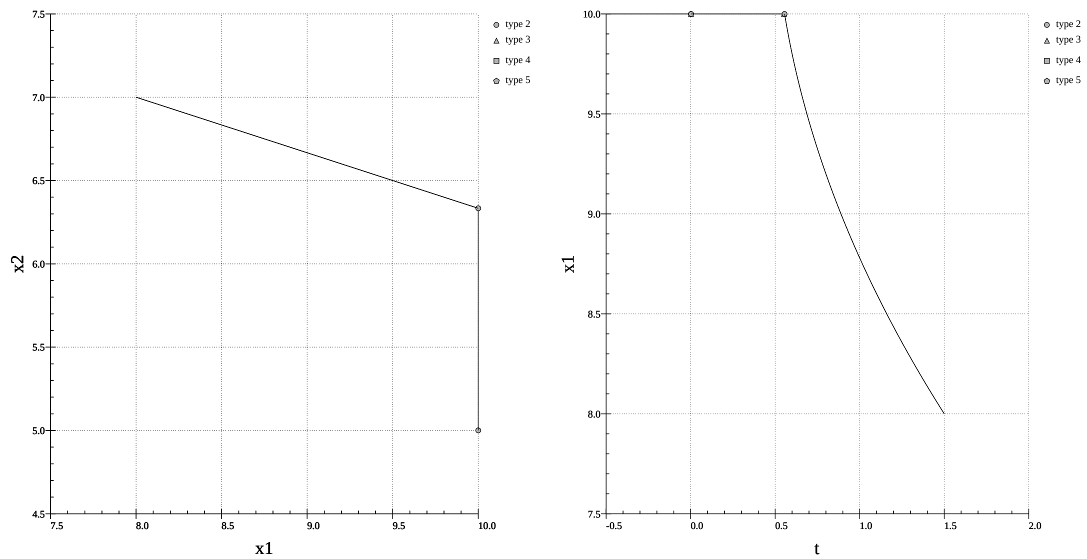



Figure 11: Example 2 in $(x_1, x_2)$ and $(t, x_1)$-space

*Proof.* Without loss of generality we consider the problem $P_1$. The proof for $P_2$ is readily the same as the proof for $P_1$. Therefore, we omit it. Set $(J_1)_o(x,t) := \{j \in J \mid g_j(x,t) = 0, \quad j \neq s{+}3\}$. Then, for all $j \in J_o(x,t) = (J_1)_o(x,t) \cup \{s{+}3\}$, we have

$$\begin{aligned} g_j(x,t) &= 0 \Longleftrightarrow \begin{cases} g_j(x) = 0 & \text{if } j \neq s+3 \\ g_j(x) = -(t-1)g_j(x^o) & \text{if } j = s+3 \end{cases} \\ D_x g_j(x,t) &= D_x g_j(x). \end{aligned}$$

Let $x \in M_1(t)$, $j \in J_o(x,t)$. In view of Definition 2, it suffices indeed to check that the given vector $\xi$ is also an MF-vector in $(x,t)$. For $j \neq s+3$ we have $D_x g_j(x,t)\xi = D_x g_j(x)\xi < -g_j(x) = 0$ and for $j = s+3$ we obtain

$$D_x g_j(x,t)\xi = D_x g_j(x)\xi < -g_j(x) = (t-1)g_j(x^o) \leq 0, \quad \forall t \in (-\infty, 1].$$

Hence $\xi$ is an MF-vector. □

**Remark 6.** $(a)$ *In case, the MFCQ is satisfied for all* $x \in M_i(t), t \in (-\infty, 1], i = 1, 2$ *the set* $M_i(t_1)$ *is homomorphic to* $M_i(t_2)$ *for all* $t_1, t_2 \in [0,1], i = 1, 2.$

$(b)$ *Assume (C4) and that the MFCQ is satisfied for all* $x \in M_i(t), t \in (-\infty, 1]$. *Then we have* $\sum_{gc} \cap \sum_{gc}^4 = \emptyset$.

**Remark 7.** *Assume* $\bar{z} \in \sum_{gc}$ *with* $\bar{x} \in H_1$ *and* $s{+}3 \in J_o(\bar{z})$ *respectively* $\bar{x} \in H_2$ *and* $s+1 \in J_o(\bar{z})$. *Then it holds that*

$$\bar{z} \in \textstyle\sum_{gc}^4 \cup \sum_{gc}^5,$$

*and the MFCQ is not satisfied.*

According to the propositions $\mathbf{A})(i)$, $(v)$ and $\mathbf{C}$ of Theorem 2 and the example in Fig. 5, we know that points of Type 4 and 5, where the MFCQ is not satisfied, may appear.

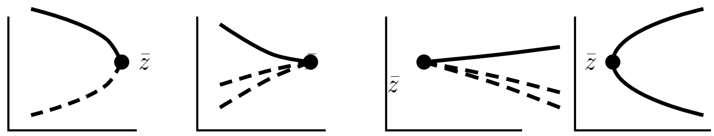
**Definition 3 (Gomez [23]).** *We assume* $P_i(t)$,$i = 1, 2$, *to be* $JJT$*-regular with respect to* $(-\infty, 1]$ *and* $\bar{z} = (\bar{x}, \bar{t}) \in \sum_{gc}$. *A g.c. point* $\bar{z}$ *is called a turning point in negative direction (positive direction) if there exists a neighbourhood* $V(\bar{z})$, *such that* $\forall (x,t) \in \sum_{gc} \cap V(\bar{z})$ *holds for* $t \leq \bar{t} \quad (t \geq \bar{t})$.

The Figures 12 $(a)$ and $(b)$ show turning points in negativ direction and the Figures $(c)$ and $(d)$ those in positive direction.

**Theorem 5.** *Assume* $\mathbf{P_i}(t)$, $i = 1, 2$, *to be* $JJT$*-regular with respect to* $(-\infty, 1]$. *Then every turning point without possibilities to jump is a turning point in negative direction.*

*Proof.* The proof relies on Theorem 1 in Gómez [23]. We will divide the proof into two steps. Without loss of generality we consider the problem $P_1$.

Step 1 (Type 5).- Let $J_o(\bar{z}) = \{1, \cdots, p\}$ and $p = s+3$. We deduce from the




$(a)$ $(b)$ $(c)$ $(d)$

Type 4 Type 5 Type 5 Type 4

Figure 12: 5 Turning points in negativ and positiv direction

properties of a point of Type 5 that the set $\{D_x g_j(\bar{z}), j = 1, \cdots, p-1\}$ is linearly independent. Then we obtain a uniquely determined solution $\bar{u}^p = (\bar{u}_1^p, \cdots, u_{p-1}^p)$ of the system

$$D_x f(\bar{z}) + \sum_{j=1}^{p-1} \bar{\mu}_j^p D_x g_j(\bar{z}) = 0$$

with $\bar{\mu}_j^p \neq 0$, $j = 1, \cdots, p-1$.
Next, consider the system

$$\begin{bmatrix} D_x f(x,t) + \sum_{j=1}^{p-1} \mu_j^p D_x g_j(x,t) \\ g_1(x,t) \\ \vdots \\ g_{p-1}(x,t) \end{bmatrix} = 0. \tag{3}$$

By the Implicit Function Theorem, there exists a uniquely determined function

$$(x^p(t), u^p(t)) : (\bar{t} - \epsilon, \bar{t} + \epsilon) \to \mathbb{R}^{n+p-1}, \tag{4}$$

such that

$(a)$ $x^p(\bar{t}) = \bar{x}, \quad u^p(\bar{t}) = \bar{u}^p,$

$(b)$ for each $t \in (\bar{t} - \epsilon, \bar{t} + \epsilon)$ the vector $(x^p(t), u^p(t), t)$ is a solution of (3).

Recall that $(x^p(t), t)$ is a g.c. point if the inequality

$$g_p(x^p(t), t) = g_{s+3}(x^p(t), t) \leq 0, \tag{5}$$

holds. Hence, the curve $(x^p(t), t)$ belongs to $\sum_{gc}$ either for $t \in (\bar{t}, \bar{t} + \epsilon)$ or for $t \in (\bar{t} - \epsilon, \bar{t})$. It suffices to show that

$$\frac{d}{dt} g_{s+3}(x^p(\bar{t}), \bar{t}) > 0,$$

since $g_p$ becomes strictly positive, i.e., $(x^p(t), t)$ is not feasible.
We calculate this quantity:

$$\frac{d}{dt} g_p(x^p(\bar{t}), \bar{t}) = D_x g_p(\bar{x}, \bar{t}) \dot{x}^p(\bar{t}) + D_t g_p(\bar{x}, \bar{t}). \tag{6}$$

In view of (4) $(a),(b)$ , we see that $\frac{d}{dt}g_j(x^p(\bar{t}),\bar{t}) = 0, j = 1,\cdots,p-1$, since $g_j$ vanishes along $(x^p(t),t)$. Hence, we have

$$D_x g_j(\bar{z})\dot{x}^p(\bar{t}) = -D_t g_j(\bar{z}) = 0, \quad j = 1,\cdots,p-1. \tag{7}$$

Since the MFCQ is not satisfied, there exist $\bar{\mu}_j > 0$, $j \in J_o(\bar{z})$, such that

$$\sum_{j=1}^{p} \bar{\mu}_j D_x g_j(\bar{z}) = 0. \tag{8}$$

Multiplying (8) by $\dot{x}^p(\bar{t})$ and combining (7), we are led to

$$-\sum_{j=1}^{p-1} \bar{\mu}_j D_t g_j(\bar{z}) + \bar{u}_p D_x g_p(\bar{z})\dot{x}^p(\bar{t}) = 0. \tag{9}$$

Since $D_t g_j(\bar{z}) = 0$, $j = 1,\cdots,p-1$ (cf. Definition of $\mathbf{P_1}$), we obtain

$$\bar{u}_p D_x g_p(\bar{z})\dot{x}^p(\bar{t}) = 0. \tag{10}$$

Hence, (6) is equivalent to

$$\frac{d}{dt}g_p(x^p(\bar{t}),\bar{t}) = D_t g_p(\bar{x},\bar{t}) = g_p(x^o) > 0,$$

which yields the desired conclusion.
Step 2 (Type 4).-The proof is readily the same as in Gómez [23]. Therefore, we recall just the main idea. We consider the following optimization problem:

$$(\hat{P}) \qquad \min\{\hat{F}(x,u^p,t,u_o) \mid (x,u^p,t,u_o) \in \hat{M}\},$$

where

$$\begin{aligned}
\hat{F}(x,u^p,t,u_o) &= t \\
\hat{M} &= \{(x,u^p,t,u_o) \in \mathbb{R}^{n+p+1} \mid \Upsilon(x,u^p,t,u_o) = 0\},
\end{aligned}$$

$$\Upsilon(x,u^p,t,u_o) = \begin{bmatrix} D_x\mathcal{L}(x,u_1,\cdots,u_{p-1},t,u_o) \\ g_1(x,t) \\ \vdots \\ g_p(x,t) \end{bmatrix}$$

and $\mathcal{L}(x,u_1,\cdots,u_{p-1},t,u_o) = u_o f(x,t) + \sum_{j=1}^{p-1} u_j g_j(x,t) + \bar{u}_p g_p(x,t)$.
It suffices to show that

$$D^2\hat{L}(\bar{x},\bar{u}^p,\bar{t},0)|_{T_{(\bar{x},\bar{u}^p,\bar{t},0)}\hat{M}} < 0,$$

where $\hat{L}$ is the Lagrangian corresponding to the problem $(\hat{P})$. In particular, considering $P_1(t)$ we have

$$D^2\hat{L}(\bar{x},\bar{u}^p,\bar{t},0)|_{T_{(\bar{x},\bar{u}^p,\bar{t},0)}\hat{M}} = \frac{w_x^T D_x^T f(\bar{z}) w_{u_o}}{u_p g_p(x^o)}, \quad p = s+3,$$

where $w = (w_x, w_{u^p}, w_t, w_{u_o}) \in \mathbb{R}^{n+p+1}$ with $D\Upsilon(\bar{x}, \bar{u}^p, \bar{t}, 0)w = 0_{n+p}$ and $w_{u_o} \neq 0$.
The conclusion follows in the same way as in [23]. $\square$

### 3.2.1 Proposal to overcome turning points without possibilities to jump

First, we observe that there is an analogy between the embeddings $P_i(t), i = 1, 2,$ and cutting plane methods. In this case the "cutting sets" are defined as the level sets of $g_{s+3}$ and $g_{s+1}$, respectively:

$$\begin{aligned} S_1(t) &= \mathcal{P} \cap \{x \mid g_{s+3}(x,t) \geq 0\} \quad t \in (-\infty, 1] \\ S_2(t) &= \mathcal{P} \cap \{x \mid g_{s+1}(x,t) \geq 0\} \quad t \in (-\infty, 1]. \end{aligned}$$

If a turning point $(\bar{x}, \bar{t})$ occurs without jump possibilities, we cannot "cut" the set $S_1(\bar{t})$ or $S_2(\bar{t})$ from $\mathcal{P}$. However there exists a $\hat{t} > \bar{t}$ such that $S_1(\hat{t})$ and $S_2(\hat{t})$, respectively can be cut from $\mathcal{P}$ (see Fig. 13 ). We *try to find a* $\hat{t}$, in such a way that a cut is possible. We propose to solve the following problem:

## Dynamic choice of a starting point

Let $(\bar{x}, \bar{t})$ be a turning point without jump possibilities. We compute a local, stationary or a g.c. point $(\tilde{x}, \tilde{t}) \neq (\bar{x}, \bar{t})$ (cf. Fig. 13) of

$$P^i_{\mathbf{h}}: \qquad \max\{t \mid x \in M^i_h\} \quad i = 1, 2,$$

where

$$M_h^1 := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ccc} x & \in & \mathcal{P} \\ g_{s+1}(x,t) & \leq & 0 \\ g_{s+2}(x,t) & \leq & 0 \\ g_{s+3}(\bar{x},\bar{t}) - g_{s+3}(x,t) & \leq & 0, \\ ||x-\bar{x}||^2 & \leq & \alpha \end{array} \right\},$$

and

$$M_h^2 := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ccc} x & \in & \mathcal{P} \\ g_{s+1}(\bar{x},\bar{t}) - g_{s+1}(x,t) & \leq & 0, \\ ||x-\bar{x}||^2 & \leq & \alpha \end{array} \right\},$$

respectively, and the number $\alpha$ is sufficiently large.
If the problem $P^i_{\mathbf{h}}$, $i = 1, 2$ has a solution, we may consider two cases (w.l.o.g, we consider the problem $P_1$):
<u>*case I*</u> : $\tilde{x} \in M_1$
Then we are finished because $\tilde{x}$ is a feasible point for $P_1$.

<u>*case II*</u>: $g_{s+3}(\tilde{x}) > 0$ and $\tilde{x} \in int(\mathcal{P} \cap G_1 \cap G_2)$
We define a new starting point by setting $x^o := \tilde{x}$.

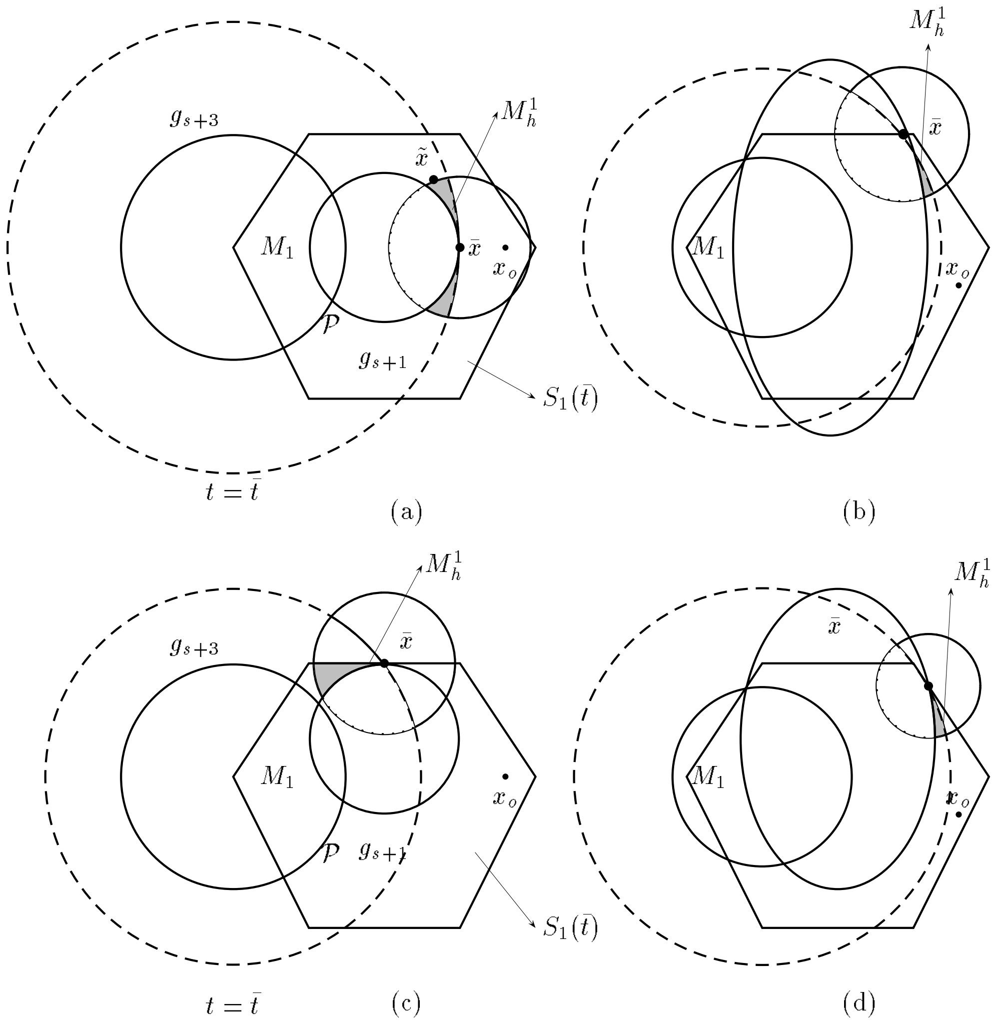



Figure 13: Trial to overcome a turning point without jump possibility

<u>*case III*</u>: $g_{s+3}(\tilde{x}) > 0$ and $\tilde{x} \in \partial\mathcal{P} \cup \partial G_1 \cup \partial G_2$
Compute a direction of descent $\tilde{d}$ such that

$$\begin{aligned} g_j(\tilde{x} + \lambda\tilde{d}) &< g_j(\tilde{x}) \quad j \in J_o(\tilde{x}) \\ \tilde{x} + \lambda\tilde{d} &\in M_h^1, \quad \lambda \in \mathbb{R}. \end{aligned}$$

If $\tilde{d}$ exists, we set

$$x^o = \tilde{x} + \lambda\tilde{d}$$

($\lambda$ sufficiently small). Otherwise, we increase the value of $\alpha$ and solve $P_h^i$. Then we proceed as in the cases I-III. We can now formulate the following algorithm for determining a g.c. point of $P_i$:

**Algorithm** 1

*Step* 0 $\alpha_{max} = r$, $r \in \mathbb{R}_+$.

*Step* 1 Choose $x^o \in \mathbb{R}^n$ such that $(C2)$ is fulfilled.

*Step* 2 Compute a connected path with the algorithm PATH*III*. If case of no jump, go to step 3. Else, proceed applying JUMP II.

*Step* 3 If $\alpha > \alpha_{max}$, stop (the solution could not be found). Else, compute an $\tilde{x}$ solving $(P_h^i)$ and go to step 4.

*Step 4* If $\tilde{x} \in M_i$, $i = 1, 2$, stop. If $(C2)$ is fulfilled, set $x^o = \tilde{x}$ and go to step 2. Else, compute a direction of descent $d$ and a number $\lambda$ such that $\tilde{x} + \lambda\tilde{d} \in M_h^i$. If $d$ exists, set $x^o := \tilde{x} + \lambda\tilde{d}$ and go to step 2. Else, increase the value of $\alpha$ and go to step 3.

# 4 Regularization

In this section we answer the question "under which conditions" the considered problems $P_i(t)$, $i = 1, 2$, are JJT-regular. This is a justification of the condition (C4). Recall that there is another kind of justification (see for example the case of the penalty-embedddings [23]), which we will not consider in this paper.

**Theorem 6.** *Assume that $\mathcal{A}^n$ is the set of all symmetric (n,n)-matrices. Let $A, B_i \in \mathcal{A}^n \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$, for $i = 1, \cdots, 3$, $a_j \in \mathbb{R}^n$, for $j = 1, \cdots, s+3$ and $b := (b_1, \cdots, b_{s+3} := g_{s+3}(x^o))^T \in \mathbb{R}^{s+3}$. We take $\mathcal{B} := (a_1, \cdots, a_{s+3}, b)$, where $s$ is the number of linear constraints. Consider next the perturbed problem defined by*

$$P_{I_{(A,\mathcal{B})}}(t): \qquad \min\left\{(x-x^o)^T A(x-x^o) | x \in M_I(\mathcal{B}, t)\right\}, \quad \textit{where}$$

$M_I(\mathcal{B}, t) :=$

$$\left\{ x \in R^n \left| \begin{array}{rcll} g_j(x,t,a_j,b_j) & := & a_j^T x + b_j & \leq 0 \\ g_{s+1}(x,t,a_{s+1},b_{s+1}) & := & x^T B_1 x + a_{s+1}^T x + b_{s+1} & \leq 0 \\ g_{s+2}(x,t,a_{s+2},b_{s+2}) & := & x^T B_2 x + a_{s+2}^T x + b_{s+2} & \leq 0 \\ g_{s+3}(x,t,a_{s+3},b_{s+3}) & := & x^T B_3 x + a_{s+3}^T x + (t-1) b_{s+3} & \leq 0 \end{array} \right. \right\},$$

$$j \in J := \{1, \cdots, s+3\} \quad j \neq s+1, \cdots, s+3.$$

*Then, for almost all $(A, \mathcal{B}) \in \mathcal{A}^n \times \mathbb{R}^{(s+3)(n+1)}$ the problem $P_{I_{(A,\mathcal{B})}}(t)\Big|_{t \neq 1, t \in R}$ is* JJT-regular *(i.e., $P_{I_{(A,\mathcal{B})}}(t)\Big|_{t \neq 1, t \in R} \in \mathcal{F}$).*

*„For almost all" means: every Lebesgue measurable subset of*

$$\{(A, \mathcal{B}) \in \mathcal{A}^n \times \mathbb{R}^{(s+3)(n+1)} \mid P_{I_{(A,\mathcal{B})}}(t) \notin \mathcal{F}\}$$

*has the Lebesgue-measure zero.*

Since the proof runs with the analogous techniques as the proof of the Theorem 5.1 in Rückmann, Tammer [21], we only present its main idea.

**Main idea of the proof:**
**Step 1**: Prove that for almost all $(A, \mathcal{B}) \in \mathcal{A}^n \times \mathbb{R}^{(s+3)(n+1)}$, each critical point of $P_{I_{(A,\mathcal{B})}}(t)$ that satisfies the (LICQ) is either a point of Type 1 or Type 2 or Type 3.
**Step 2**: Show that for almost all $(A, \mathcal{B}) \in \mathcal{A}^n \times \mathbb{R}^{(s+3)(n+1)}$ the set of those feasible points that do not satisfy the (LICQ) is the union of finitely many zero-dimensional manifolds. We also show that each of these points is either of Type 4 or Type 5, for any fixed point of these $(A, \mathcal{B})$.
We begin with some preliminary results, which will enter into the proof.

**Preliminary results**

**Lemma 1.** *(cf. [21])*
*Assume that $\mathcal{M}^n \subset \mathbb{R}^{\frac{1}{2}n(n+1)}$ is the set of all symmetric (n,n)-matrices. Further, let $\bar{I} \subset \{1, \cdots, n\}$. Next define*

$$\mathcal{A}^n(\bar{I}) := \left\{ A \in \mathcal{M}^n \;\middle|\; \begin{array}{l} rank(A) = Card(\bar{I}) \text{ and the columns of } A \\ \text{whose indices belong to } \bar{I} \text{ are linearly independent} \end{array} \right\}.$$

*Then, $\mathcal{A}^n(\bar{I})$ is a smooth manifold with*

$$Cod(\mathcal{A}^n(\bar{I}) = \frac{1}{2}(n - Card(\bar{I}))(n - Card(\bar{I}) + 1)),$$

where „Cod“ denotes the codimension.

**Lemma 2.**
*Let $g_j \in C^k(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $j \in J_o(x,t,\mathcal{B}) := \{j \in J \mid g_j(x,t,\mathcal{B}) = 0\}$ and $|J_o(x,t,\mathcal{B})| = p$. $\mathcal{B}$ and $M_I(\mathcal{B},t)$ are defined as above.*

$$M(\mathcal{B}) := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \begin{array}{l} x \in M(\mathcal{B},t) \\ \text{and the vectors } D_x g_j(x,t,a_j,b_j), j \in J_o(x,t,\mathcal{B}) \\ \text{are linearly dependent, i.e., the (LICQ)} \\ \text{is not satisfied} \end{array} \right\}.$$

*Then, for almost all $\mathcal{B}$, the following two conditions are fulfilled:*

*(a) The set $M(\mathcal{B})$ is a zero-dimensional manifold.*

*(b) For every point $(\bar{x}, \bar{t}) \in M(\mathcal{B})$ it holds that*

$$dim\ span\ \{D_{(x,t)} g_j(x,t,a_j,b_j), j \in J_o(x,t,\mathcal{B})\} = p.$$

*Proof.* Let $J^p \subset J \neq \emptyset$ with

$$J^p = \{j_1, \cdots, j_p\}, j_i \in \{1, \cdots, s+3\}, i = 1, \cdots, p. \tag{11}$$

W.l.o.g, we set $j_p = s+3$ $(p \leq s+3)$. For $J^p$ the following cases may occur:

**case** 1 $J^p = I_o^{p-1}(\bar{x},\bar{t}) \cup \{s+3\}$,

**case** 2 $J^p = I_o^{p-2}(\bar{x},\bar{t}) \cup \{s+2, s+3\}$,

**case** 3 $J^p = \{s+2, s+3\}$,

**case** 4 $J^p = I_o^{p-2}(\bar{x},\bar{t}) \cup \{s+1, s+3\}$,

**case** 5 $J^p = I_o^{p-3}(\bar{x},\bar{t}) \cup \{s+1, s+2, s+3\}$,

**case** 6 $J^p = \{s+1, s+2, s+3\}$,

where $I_o^k(\bar{x},\bar{t})$ denotes the active linear constraints (cf. (2)). We may identify the set $J^p$ with $\{1,\cdots,p\}$. Then, from (11), we have $p = s+3$.

Let $\hat{q}_p \in \mathbb{R}$, $\hat{q}_p \neq 0, q \in \mathbb{R}^{p-1}$. We define

$$\tilde{\mathcal{L}}(x,q,t,\mathcal{B}) := \sum_{j=1}^{p-1} q_j g_j(x,t,a_j,b_j) + \hat{q}_p g_p(x,t,a_p,b_p) \tag{12}$$

$$\tilde{G}(x,q,t,\mathcal{B}) := \begin{bmatrix} D_x\tilde{\mathcal{L}}(x,q,t,\mathcal{B}) \\ g_j(x,t,a_j,b_j), j \in J^p \end{bmatrix}. \tag{13}$$

It suffices to consider the case 1. Later we will see that, for other cases, the proof can be completed in a similar way. Since the partial derivative of $\tilde{G}$,

$$D_{(a_p,b_1,\cdots,b_p)}\tilde{G}(x,q,t,\mathcal{B}) = \left[\begin{array}{cccc} \frac{\partial}{\partial a_p} & \frac{\partial}{\partial b_1} & \cdots & \frac{\partial}{\partial b_p} \\ \hline \hat{q}_p I_n & 0 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ x^T & & & t-1 \end{array}\right] = \begin{bmatrix} \hat{q}_p I_n & & 0 \\ 0 & & \\ & I_{p-1} & \\ x^T & & t-1 \end{bmatrix},$$

is a regular matrix for all $t \neq 1$ ($I_n$, $I_{p-1}$ denote the identity matrices), $0 \in \mathbb{R}^{n+p}$ is a regular value of $\tilde{G}$, for $t \neq 1$. Using $\tilde{G} \in C^1(\mathbb{R}^{n+p} \times \mathbb{R}^{p*(n+1)}, \mathbb{R}^{n+p})$ and the parametrized Sard's Theorem, it follows for almost all $\mathcal{B}$ that $0 \in \mathbb{R}^{n+p}$ is a regular value of

$$\tilde{G}_{\mathcal{B}}(x,q,t) := \tilde{G}(x,q,t,\mathcal{B}) \tag{14}$$

and thus, the set $\tilde{G}_{\mathcal{B}}^{-1}(0)$ and its projection in the $(x,t)$-space

$$\pi_{(x,t)}(\tilde{G}_{\mathcal{B}}^{-1}(0)) := \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \begin{array}{l} g_j(x,t,a_j,b_j) = 0, j \in J^p \\ D_x g_p(x,t,a_p,b_p) \in \text{span } \{D_x g_j(x,t,a_j,b_j)\} \\ j = 1,\cdots,p-1 \end{array} \right\}$$

are zero-dimensional manifolds, since $(x,q,t) \in \mathbb{R}^{n+p}$ and the system $\tilde{G}(x,q,t,\mathcal{B}) = 0$ contains $n+p$ inequalities. $(b)$ follows from $(a)$, since for almost all $\mathcal{B}$, 0 is a regular value of $\tilde{G}$. Hence, for each point $(\bar{x},\bar{t}) \in \pi_{(x,t)}(\tilde{G}_{\mathcal{B}}^{-1}(0)) = M(\mathcal{B})$ we have

$$\dim \operatorname{span} \{D_{(x,t)}g_j(x,t,a_j,b_j), j \in J_o(x,t,\mathcal{B})\} = p.$$

For other cases, the conclusion will be similar, since the partial derivatives do not depend on the numbers $b_1, \cdots, b_p$ and the vector $a_p$. □

**Proof of Theorem 6**

*Proof* (Step 1)
Let $J^p \subset J$ and w.l.o.g we take $J^p$ as defined in case 1. We easily verify that the objective function can be given by

$$(x - x^o)^T A(x - x^o) = x^T Ax + c^T x + {x^o}^T Ax^o, \tag{15}$$

with $c := -2Ax^o$.
Next let $\nu = 0, 1$ and define

$$\begin{aligned} A^0(z) &= (D_x^T g_1(z), \cdots, D_x^T g_{p-1}(z)) = (a_1, \cdots, a_{p-1}) \\ A^1(z) &= \left(D_x^T g_1(z), \cdots, D_x^T g_p(z)\right) = (a_1, \cdots, 2B_3x + a_{s+3}) \end{aligned}$$

and

$$M^\nu(x,\mu,t) = \begin{pmatrix} D_x^2 f(x,t) + \sum_{j=1}^p \mu_j D_x^2 g_j(x,t) & A^\nu(x,t) \\ -A^\nu(x,t)^T & 0 \end{pmatrix} \tag{16}$$

$$H^o(x,\mu,t) = \begin{bmatrix} D_x f(x,t) + \sum_{j=1}^p \mu_j D_x g_j(x,t) \\ -g_1(x,t) \\ \vdots \\ -g_{p-1}(x,t) \\ \mu_p - g_p(x,t) \end{bmatrix} \tag{17}$$

$$H^1(x,\mu,t) = \begin{bmatrix} D_x f(x,t) + \sum_{j=1}^p \mu_j D_x g_j(x,t) \\ -g_1(x,t) \\ \vdots \\ -g_{p-1}(x,t) \\ -g_p(x,t) \end{bmatrix}. \tag{18}$$

For the perturbed problem with $a = (A, \mathcal{B}, c)$, we have
$H_a^0(x,\mu,t) :=$

$$\begin{bmatrix} 2Ax + c + \sum_{j=1}^k \mu_j a_j + \mu_{s+3}(B_3x + a_{s+3}) \\ -(a_1^T x + b_1) \\ \vdots \\ \mu_p - (x^T B_3 x + a_{s+3}^T x + (t-1)g_{s+3}(x^o)) \end{bmatrix}$$

$H_a^1(x,\mu,t) :=$

$$\begin{bmatrix} 2Ax + c + \sum_{j=1}^{k} \mu_j a_j + \mu_{s+3}(B_3 x + a_{s+3}^T) \\ -(a_j^T x + b_1) \\ \vdots \\ -(x^T B_3 x + a_{s+3}^T x + (t-1) g_{s+3}(x^o)) \end{bmatrix}$$

$$M_a^0(x,\mu,t) := \begin{bmatrix} 2A + \mu_{s+3} B_3 & A^0(x,t) \\ -A^0(x,t)^T & 0 \end{bmatrix}$$

$$M_a^1(x,\mu,t) := \begin{bmatrix} 2A + \mu_{s+1} B_1 + \mu_{s+3} B_3 & A^1(x,t) \\ -A^1(x,t)^T & 0 \end{bmatrix}.$$

Now, we define the following sets

(19)
$$\mathcal{M}(I) := \mathbb{R}^{n+p+1} \times O_{n+p} \times \mathcal{A}^{n+p}(I),$$

(20) $$\mathcal{M}_a^\nu := \{\mathcal{Z} \in \mathbb{R}^{\tilde{q}} \mid \mathcal{Z}^2 = H_a^\nu(\mathcal{Z}^1), \mathcal{Z}^3 = M_a^\nu(\mathcal{Z}^1)\}, \qquad \nu = 0, 1,$$

where

$$\begin{aligned} \tilde{q} &= (n+p+1) + (n+p) + \frac{1}{2}(n+p)(n+p+1) \\ \mathcal{Z} &= (\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3) \in \mathbb{R}^{\tilde{q}} \\ \mathcal{Z}^1 &= (x,\mu,t) \in \mathbb{R}^{n+p+1} \\ \mathcal{Z}^2 &\in \mathbb{R}^{n+p} \\ \mathcal{Z}^3 &\in \mathbb{R}^{\frac{1}{2}(n+p)(n+p+1)} \end{aligned}$$

with $I \subset \{1, \cdots, n+p\}$, and for any index set $\tilde{I} \subset \{1, \cdots, p\}$ we consider the set

(21)
$$\mathcal{M}(I, \tilde{I}) := \mathcal{M}(I) \cap \{\mathcal{Z} \in \mathbb{R}^{\tilde{q}} \mid \mathcal{Z}_{n+j} = 0, j \in \tilde{I}\}, \text{with } \mathcal{Z}_{n+j} = \mu_j, j \in \tilde{I}.$$

**Lemma 3.**
*The sets $\mathcal{M}_a^\nu, \mathcal{M}(I)$ and $\mathcal{M}(I, \tilde{I})$ are $C^1$-manifolds.*

*Proof of Lemma.* Taking into account the equations describing all these sets, we have:

- For $\mathcal{M}_a^\nu, \quad \nu = 0, 1:$

(22)

| equality | $\frac{\partial}{\partial \mathcal{Z}^1}$ | $\frac{\partial}{\partial \mathcal{Z}^2}$ | $\frac{\partial}{\partial \mathcal{Z}^3}$ |
|---|---|---|---|
| $\mathcal{Z}^2 - H_a^\nu = 0$ | $\circledast$ | $I_{n+p}$ | $0$ |
| $\mathcal{Z}^3 - M_a^\nu = 0$ | $\circledast$ | $0$ | $I_{\frac{1}{2}(n+p)(n+p+1)}$ |

- For $\mathcal{M}(I)$: from Lemma (1), we have

$$
(23)\qquad
\begin{array}{cccc}
\text{equality} & \frac{\partial}{\partial \mathcal{Z}^1} & \frac{\partial}{\partial \mathcal{Z}^2} & \frac{\partial}{\partial \mathcal{Z}^3} \\
\hline
\mathcal{Z}^2 = 0 & 0 & I_{n+p} & 0 \\
\circledast & 0 & 0 & \left( I_{\frac{1}{2}(n+p-|I|)(n+p-(|I|+1))} \mid \circledast \right).
\end{array}
$$

Recall that $Cod\mathcal{A}^{n+p}(I) = \frac{1}{2}(n+p- \mid I \mid)(n+p-(\mid I \mid +1))$ and $\circledast$ denotes any matrix without importance for our analysis. Since the rows of the matrix (23) are linearly independent, we obtain the desired conclusion. The codimensions are given by

$$
\begin{aligned}
\text{Cod}(\ \mathcal{M}_a^\nu) &= n+p+\frac{1}{2}(n+p)(n+p+1) \\
&= \tilde{q} - (n+p+1) \\
\text{Cod}(\ \mathcal{M}(I)) &= n+p+\text{Cod }(\mathcal{A}^{n+p}(I)) \\
&= n+p+\frac{1}{2}(n+p-\text{Card}(I))(n+p-\text{Card}(I)+1).
\end{aligned}
$$

- $\mathcal{M}(I,\tilde{I})$ is described by all describing equalities for $\mathcal{M}(I)$ and:

$$
\begin{array}{cc}
\text{equality} & D_{z_{n+j}}(j \in \tilde{I}) \\
\hline
z_{n+j} = 0 & I_{|\tilde{I}|}
\end{array}.
$$

  The conclusion follows in similar ways as above. The codimension is given by

$$
\text{Cod}(\mathcal{M}(I,\tilde{I})) = \text{Cod}(\mathcal{M}(I)) + \mid \tilde{I} \mid .
$$

Finally , we consider the enlarged jacobian of all describing equalities of $\mathcal{M}_a^\nu, \mathcal{M}(I)$, $\mathcal{M}(I,\tilde{I})$, and the partial derivatives with respect to $(A,b,c)$, $b=(b_1,\cdots,b_p)$, $c=-2Ax^o$ (cf. (15)).

(24)

| | $D_{\mathcal{Z}^1}$ | $D_{\mathcal{Z}^2}$ | $D_{\mathcal{Z}^3}$ | $D_A$ | $D_b$ | $D_c$ |
|---|---|---|---|---|---|---|
| $\mathcal{Z}^2 - H_a^\nu(\mathcal{Z}^1) = 0$ | $*$ | $I^{n+p}$ | $0$ | $\begin{matrix} * \\ * \end{matrix}$ | $\begin{matrix} 0 \\ I_p(t) \end{matrix}$ | $\begin{matrix} -I^n \\ 0 \end{matrix}$ |
| $\mathcal{Z}^3 - M_a^\nu(\mathcal{Z}^1) = 0$ | $*$ | $0$ | $I^{\frac{1}{2}(n+p)(n+p+1)}$ | $\begin{matrix} -2I^{\frac{1}{2}n(n+1)} & 0 \\ 0 & 0 \end{matrix}$ | $\begin{matrix} 0 \\ 0 \end{matrix}$ | $\begin{matrix} 0 \\ 0 \end{matrix}$ |
| $\mathcal{Z}^2 = 0$ | $0$ | $I^{n+p}$ | $0$ | $\begin{matrix} 0 & 0 \end{matrix}$ | $0$ | $0$ |
| $\mathcal{A}^{n+p}(I)$ | $0$ | $0$ | $-I^{\lambda_1} \mid *$ | $0$ | $0$ | $0$ |
| $z_{n+j}$ | $I^{\lvert\tilde{I}\rvert} \mid 0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

where

$$I_p(t) = \begin{bmatrix} -1 & 0 & 0 \\ & \ddots & 0 \\ & & -(1-t) \end{bmatrix}$$

is a $(p,p)-$matrix and $\lambda_1 = \frac{1}{2}(n+p- \mid I \mid)(n+p- \mid I \mid +1)$.
The matrix (24) has full rank for all $\{n+1,\cdots,n+p\} \in I$ and $t \neq 1$.
We recall that for other cases we have the same result, since the number of linearly independent rows in (22), (23) and also the partial derivatives w.r.t $(A,b,C)$ in (24) do not depend on the active constraints.

Next, let $(x,t)$ be a g.c. point of $P_{I_{(A,\mathcal{B})}}(t)$ satisfying the LICQ with $J_o(x,t,\mathcal{B}) = \{1,\cdots,p\}$ and $H_a^1(x,\mu,t) = 0$. Obviously, there exists an index set $I \subset \{1,\cdots,n+p\}$ satisfying $\{n+1,\cdots,n+p\} \subseteq I$ with $M_a^1(x,\mu,t) \subset \mathcal{A}(I)$. Recall that $\mu$ is uniquely determined. For each $(I,\tilde{I})$, with $\{n+1,\cdots,n+p\} \subset I$, the matrix (24) has full rank. Since $|I|$ is finite, using the parametrized Sard's Theorem, the following holds:

$$\mathcal{M}(I,\tilde{I}) \pitchfork \mathcal{M}_a^\nu, \quad \nu = 0,1, \quad \tilde{I} \subset \{1,\cdots,p\}.$$

On the other hand,

(25)

$$\begin{aligned}
\dim \mathcal{M}(I,\tilde{I}) \cap \mathcal{M}_a^\nu &= \tilde{q} - \text{Cod } (\mathcal{M}(I,\tilde{I}) \cap \mathcal{M}_a^\nu) \\
&= \tilde{q} - \left[\text{Cod}(\mathcal{M}(I,\tilde{I}) + \text{Cod } \mathcal{M}_a^\nu)\right] \\
&= \tilde{q} - \text{Cod } \mathcal{M}(I) - |\tilde{I}| - \text{Cod } \mathcal{M}_a^\nu) \\
&= n+p+1 - \text{Cod } M(I) - |\tilde{I}| \\
&= n+p+1 - \left[n+p+\frac{1}{2}(n+p-|I|)(n+p-|I|+1)\right] - |\tilde{I}| \\
&= \begin{cases} 1 & \text{if } |\tilde{I}| = 0 \text{ und } |I| = n+p \\ 0 & \text{if } |\tilde{I}| = 0 \text{ und } |I| = n+p-1 \\ 0 & \text{if } |\tilde{I}| = 1 \text{ und } |I| = n+p \end{cases}
\end{aligned}$$

$$M(I,\tilde{I} \cap \mathcal{M}_a^\nu) = \emptyset \text{ for } \frac{1}{2}(n+p-|I|)(n+p-|I|+1) - |\tilde{I}| > 1,$$

Since the LICQ in $(x,t)$ is fulfilled, there exists an $I$ such that $\{n+1,\cdots,n+p\} \subset I \subset \{1,\cdots,n+p\}$, and a vector with

$$((x,\mu,t), H_a^1(x,\mu,t), M_a^1(x,\mu,t)) \in M(I,\tilde{I}) \cap \mathcal{M}_a^\nu.$$

From (25), we distinguish the following cases for $(x,t)$:

1. Card$(I) = n+p$ and $|\tilde{I}| = 0$ , i.e., $rank(M_a^\nu(x,\mu,t)) = n+p$ and $\mu_j \neq 0, \quad j = 1,\cdots,p$. Hence $(x,t)$ is a g.c. point of Type 1.

2. $\text{Card}(I) = n+p-1$ and $|\tilde{I}| = 0$, i.e. $rank(M_a^\nu(x,\mu,t)) = n+p-1$ . Then, $(x,t) \in \sum_{gc}^3$.

3. $|\tilde{I}| = 1$, i.e., $|J_o(x,a,b,t) \setminus J_+(x,a,b,t)| = 1$ and $M_a^\nu(x,\mu,t) = n+p-1$. Then, $(x,t) \in \sum_{gc}^2$.

*Proof* (Step 2) We distinguish two cases:
*Case 1* ($p > n$): In view of Lemma 2($b$), we have $p = n+1$, since $M(\beta) \subset \mathbb{R}^{n+1}$. Hence $\bar{z} \in \sum_{gc}^5$.

*Case 2* ($p <= n$): The proof of this case is readly the same as the proof in [21] (see also Mbunga [18]). Therefore, we omit it here.

The conclusion of the theorem follows from Fubini's Theorem.

# 5 Some applications

In this chapter we consider two applications: the general quadratic optimization problem and a special multi-objective optimization problem. It will be shown that pathfollowing methods with jumps can be useful for the considered applications, but not successful in any case. This is no surprise, since we do not have jumps in all situations that can appear in the set $\sum_{gc}$. Otherwise, the problem of the global quadratic optimization would be solved.

## 5.1 The general quadratic optimization problem

We consider the problem

$$QOP: \qquad \text{glob}\min\{f(x) = x^TQx + c^Tx | x \in \mathcal{P}\},$$

where $Q$ is a symmetric (n,n)-matrix and

$$\mathcal{P} = \{x \in R^n | Ax \leq b, x \geq 0\} \neq \emptyset$$

compact. According to the properties of $Q$, we distinguish three cases. We are only interested in two cases (i.e., non-convex problems).

### 5.1.1 Indefinite quadratic optimization problem

In this case the matrix $Q$ is indefinite. Then we can find positive semi-definite matrices $Q_i$, $i = 1, 2$, such that

$$Q = Q_1 - Q_2. \tag{26}$$

**Remark 8.** *We remark that (26) can be obtained by setting* $Q = (Q + \varrho I_n) - \varrho I_n$, *where* $\varrho \geq | \lambda_i |$, $i = 1, \cdots, n$, *and* $\lambda_i$ *is a proper value of* $Q$ *and* $I_n$ *the identity.*

Next, we define

$$\begin{aligned} f_1(x) &:= x^T Q_1 x + c^T x \\ f_2(x) &:= x^T Q_2 x. \end{aligned}$$

Then we have the following equivalent problem

$$IQOP: \quad \operatorname{glob\,min}\{f_I(x, x_{n+1}) \mid (x, x_{n+1}) \in M_I\},$$

where

$$\begin{aligned} f_I(x, x_{n+1}) &:= f_1(x) - x_{n+1} \\ M_I &:= \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid g_j(x, x_{n+1}) \leq 0, j \in J\} \\ J &:= \{1, \cdots, m, m+1, \cdots, m+n+1\} \\ g_j(x, x_{n+1}) &:= a_j^T x + b_j, \qquad j \in \{1, \cdots, m\} \\ g_{m+i}(x, x_{n+1}) &:= -x_i, \qquad i \in \{1, \cdots, n\} \\ g_{s+1}(x, x_{n+1}) &:= x_{n+1} - f_2(x), \qquad s := m + n. \end{aligned}$$

Recall that $f, g_j, j = 1, \cdots, s$ are convex and $g_{s+1}$ is concave.

**Theorem 7.** *(QOP) and (IQOP) are equivalent in the following sense:*

(a) *If* $\bar{x} \in \psi_{glob}(QOP)$ *(*$\psi_{loc}(QOP)$ *and* $\psi_{stat}(QOP)$*, respectively ) then* $(\bar{x}, f_2(\bar{x})) \in \psi_{glob}(IQOP)$ *(*$\psi_{loc}(IQOP)$ *and* $\psi_{stat}(IQOP)$ *respectively)*

(b) *If* $(\bar{x}, \bar{x}_{n+1}) \in \psi_{glob}(IQOP)$ *(*$\psi_{loc}(IQOP)$ *and* $\psi_{stat}(IQOP)$*, respectively), then* $\bar{x} \in \psi_{glob}(QOP)$ *(*$\psi_{loc}(QOP)$ *and* $\psi_{stat}(QOP)$*, respectively).*

*Proof.* We only show that

$(a1)$ $\bar{x} \in \psi_{glob}(QOP) \Rightarrow (\bar{x}, f_2(\bar{x})) \in \psi_{glob}(IQOP)$,

$(b1)$ $(\bar{x}, \bar{x}_{n+1}) \in \psi_{glob}(IQOP) \Rightarrow \bar{x} \in \psi_{glob}(QOP)$,

$(a3)$ $\bar{x} \in \psi_{stat}(QOP) \Rightarrow (\bar{x}, f_2(\bar{x})) \in \psi_{stat}(IQOP)$,

$(b3)$ $(\bar{x}, \bar{x}_{n+1}) \in \psi_{stat}(IQOP) \Rightarrow \bar{x} \in \psi_{stat}(QOP)$.

The local cases in $(a)$ and $(b)$ are shown in a similar way. Therefore, we omit them (cf. Mbunga [18]).

$(a1)$ $(\bar{x}, f_2(\bar{x}))$ is feasible for $(IQOP)$ and it holds that

$$f_1(\bar{x}) - f_2(\bar{x}) = f(\bar{x}) \leq f(x) \qquad \forall x \in P, \tag{27}$$

since $\bar{x} \in \psi_{glob}(QOP)$. From (27) we have

$$f_I(\bar{x}, f_2(\bar{x})) \leq f(x). \tag{28}$$

On the other hand,

$$\begin{aligned} f(x) &= f_1(x) - x_{n+1} + x_{n+1} - f_2(x) \\ &\leq f_1(x) - x_{n+1} \\ &= f_I(x, x_{n+1}), \qquad \forall (x, x_{n+1}) \quad x_{n+1} \leq f_2(x). \end{aligned}$$

From (28) and the latter inequality, we deduce that

$$f_I(\bar{x}, f_2(\bar{x})) \leq f_I(x, x_{n+1}) \qquad \forall (x, x_{n+1}) \in M_I,$$

and the conclusion follows.

$(b1)$ Since $(x, x_{n+1}) \in \psi_{glob}(IQOP)$, we have

(29) $$f_1(\bar{x}) - \bar{x}_{n+1} \leq f_1(x) - x_{n+1}, \quad \forall x_{n+1} \leq f_2(x).$$

Next we observe that

$$\begin{aligned} f(\bar{x}) &= f_1(\bar{x}) - f_2(\bar{x}) \\ &= f_1(\bar{x}) - \bar{x}_{n+1} + \bar{x}_{n+1} - f_2(\bar{x}) \\ &\leq f_1(\bar{x}) - \bar{x}_{n+1}, \text{ since } \bar{x}_{n+1} \leq f_2(\bar{x}). \end{aligned}$$

Moreover, by (29), we have

$$f(\bar{x}) \leq f_1(x) - x_{n+1}, \qquad \forall x_{n+1} \leq f_2(x).$$

With our choice of $x_{n+1} = f_2(x)$, we obtain

$$f(\bar{x}) \leq f_1(x) - f_2(x) = f(x) \qquad \forall x \in P,$$

and, hence, the desired conclusion.

$(a3)$ Since $\bar{x} \in \psi_{stat}(QOP)$, there exist $\mu_j \geq 0$, $j = 1, \cdots, |j_o(\bar{x})| = p$, such that

(30) $$D_x f(\bar{x}) + \sum_{j=1}^{p} \mu_j D_x g_j(\bar{x}) = 0.$$

From the definition of $(QOP)$ and $(IQOP)$, we deduce that

$$J_o(\bar{x}) \subset J_o(\bar{x}, f_2(\bar{x})) \text{ and } s+1 \in J_o(\bar{x}, f_2(\bar{x})).$$

It suffices to show that there exist $\tilde{\mu}_j \geq 0$, $j = 1, \cdots, |j_o(\bar{x}, \bar{x}_{n+1})| = |j_o(\bar{x})| + 1$, such that

(31)
$$(D_x f_1(\bar{x}), -1) + \sum_{j=1}^{p} \tilde{\mu}_j \, (D_x g_j(\bar{x}), 0) + \tilde{u}_{s+1} \, (-D_x f_2(\bar{x}), 1) = 0$$

is fulfilled. With our choice

(32) $$\tilde{\mu}_j := \mu_j \quad j = 1, \cdots, p \qquad \tilde{\mu}_{s+1} := 1,$$

we obtain the desired result, since $D_x f(x) = D_x f_1(x) - D_x f_2(x)$.

$(b3)$ Let $(\bar{x}, \bar{x}_{n+1}) \in \psi_{stat}(IQOP)$. We easily verify that

$$g_{s+1}(\bar{x}, \bar{x}_{n+1}) = 0 \text{ or } s+1 \in j_o(\bar{x}, \bar{x}_{n+1}).$$

Since $s+1 \notin J_o(\bar{x}, \bar{x}_{n+1})$ implies the existence of $\tilde{\mu}_j \geq 0$, $j = 1, \cdots, |J_o(\bar{x}, \bar{x}_{n+1})| = p$, such that

$$(33) \qquad (D_x f_1(\bar{x}), -1) + \sum_{j=1}^{p} \tilde{\mu}_j \, (D_x g_j(\bar{x}), 0) = 0.$$

This provides the contradiction that $-1 = 0$.
At this point, we have shown that, for each stationary point $(\bar{x}, \bar{x}_{n+1}) \in M_I$, (31) is verified with the numbers given in (32).

□

**Remark 9.** *If $(x, x_{n+1}) \in \psi_{stat}(IQOP)$, then $g_{s+1}(x, x_{n+1}) = 0$.*

**Compactness of $M_I$**
The set $M$ may be compact, but not necessarily the set $M_I \subset \mathbb{R}^{n+1}$(cf. Fig. 14, $x_{n+1}$ not bounded).

**Example 5.1.**

$$\min\{f(x_1, x_2) = x_1^2 - x_2^2 \mid (x_1, x_2) \in M\}$$
$$M := \{0 \leq x_1 \leq 10, \quad 0 \leq x_2 \leq 10\}$$

*We set* $\quad f_1(x_1, x_2) := x_1^2 \qquad f_2(x_1, x_2) := x_2^2.$

*The transformed problem reads:*

$IQOP:$
$$\min\{f(x) = x_1^2 - x_3 \mid x \in M_I\},$$

*where*

$$M_I = \{x \in \mathbb{R}^3 \mid x_1 - 10 \leq 0, -x_1 \leq 0, x_2 - 10 \leq 0, -x_2 \leq 0, x_3 - x_2^2 \leq 0\}.$$

In order to ensure the compactness of $M_I$, we consider the additional constraint

$$g_{s+2}(x, x_{n+1}) := x_{n+1}^2 - q,$$

where $q \in \mathbb{R}_+$ is sufficiently large. Then we can assume that

$$(34) \qquad M_I^c := \{(x, x_{n+1}) \in M_I \mid g_{s+2}(x) := x_{n+1}^2 - q \leq 0\}$$

is compact.

Finally, we consider the following problem:

$IQOP_c:$
$$glob\min\{f_I(x, x_{n+1}) \mid (x, x_{n+1}) \in M_I^c\},$$

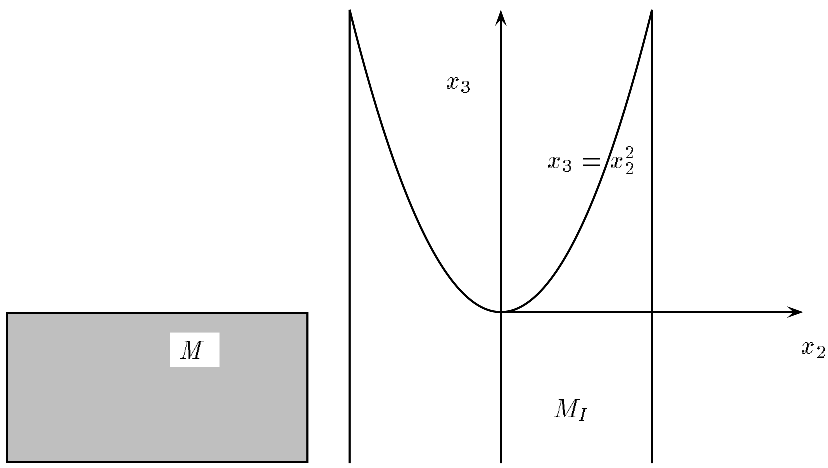



Figure 14: The feasible sets $M$ and $M_I$

where

$$
\begin{aligned}
f_I(x, x_{n+1}) &:= f_1(x) - x_{n+1} \\
f_1(x) &:= x^T Q_1 x + c^T x \\
f_2(x) &:= x^T Q_2 x,
\end{aligned}
$$

$$
\begin{aligned}
M_I^c &:= \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid g_j(x, x_{n+1}) \leq 0, j \in J\} \\
J &:= \{1, \cdots, m, m+1, \cdots, m+n+2\} \\
g_j(x, x_{n+1}) &:= a_j^T x + b_j, \qquad j \in \{1, \cdots, m\} \\
g_{m+i}(x, x_{n+1}) &:= -x_i, \qquad i \in \{1, \cdots, n\} \\
g_{s+1}(x, x_{n+1}) &:= x_{n+1} - f_2(x), \qquad s := m + n \\
g_{s+2}(x, x_{n+1}) &:= x_{n+1}^2 - q.
\end{aligned}
$$

$(QOP)$ and $(IQOP_c)$ are equivalent in the sense of Theorem 7.

We *try to solve* $(IQOP_c)$ using the strategy proposed in [5] (cf. Fig. 15).

## The strategy

- Step 1: Compute a stationary (g.c. point) $\hat{x}$ of $IQOP_c$
- Step 2: Find a point belonging to

$$M_I^\epsilon := \{(x, x_{n+1}) \in M_I^c \mid f_I(x, x_{n+1}) \leq f_I(\hat{x}, \hat{x}_{n+1}) - \epsilon\},$$

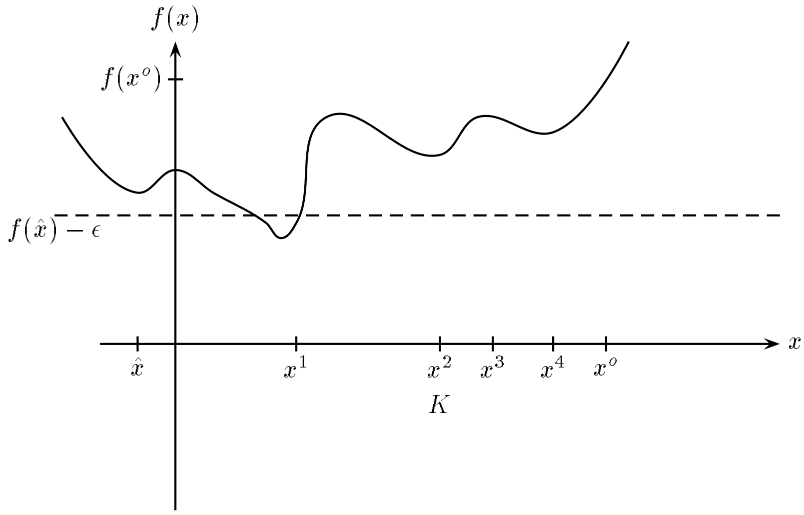



Figure 15: Step 2

with $\epsilon > 0$ sufficiently small.

Recall that the set $M_I^\epsilon$ is compact. Since the level sets of $f_I$ are closed and $\mathcal{P}$ bounded, Step 2 is the difficult one and the subject of our discussion in this section.

**Realization of Step** 2
Let $(x^o, x^o_{n+1})$ be specially choosen. Then Step 2 can be realized by finding a g.c. point of

$$\mathbf{P}_\mathbf{I}^\epsilon: \qquad \min\{f(x, x_{n+1}) := ||(x, x_{n+1}) - (x^o, x^o_{n+1})||^2 \mid (x, x_{n+1}) \in M_I^\epsilon\},$$

where

$$M_I^\epsilon := \{(x, x_{n+1}) \in M_I^\epsilon \mid g_{s+3}(x, x_{n+1}) := f_I(x, x_{n+1}) - f_I(\hat{x}, \hat{x}_{n+1}) + \epsilon \leq 0\}.$$

We introduce the following notations:

$$\tilde{A}_1 := \begin{bmatrix} -Q_2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \tilde{A}_2 := \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix} \qquad A_3 := \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad I := \{1, \cdots, n\}$$

$$\tilde{a}_j := \begin{bmatrix} a_j \\ 0 \end{bmatrix} \qquad \tilde{a}_{m+i} := -\begin{bmatrix} e_i \\ 0 \end{bmatrix} \qquad \tilde{a}_{s+1} := e_{n+1} \quad j = 1, \cdots, m; i \in I$$

$$\tilde{a}_{s+2} := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad a_{s+3} := \begin{bmatrix} c \\ -1 \end{bmatrix} \qquad \tilde{b}_j := b_j \quad j \in \{1, \cdots, m\}$$

$$\tilde{b}_{m+i} = 0, i = 1, \cdots, n \quad \tilde{b}_{s+1} := 0 \qquad \tilde{b}_{s+2} = -q \quad \tilde{b}_{s+3} := -f_I(\hat{x}, \hat{x}_{n+1}) + \epsilon,$$

where $e_i$ is the unity vector in $\mathbb{R}^{n+1}$ and $s$ as defined above. Then, the problem $P_I^\epsilon$ is equivalent to

$$\mathbf{P_I^\epsilon}: \quad \begin{aligned} &\min\{f(x) = \|x - x^o\|^2 \mid x \in M_I^\epsilon\} \\ \tilde{M}_I^\epsilon &:= \{x \in \mathbb{R}^{n+1} \mid g_j(x) \le 0, j \in J\} \\ J &:= \{1, \cdots, s\} \cup \{s+1, s+2, s+3\} \\ g_j(x) &:= \tilde{a}_j^T x + \tilde{b}_j, \quad j \in \{1, \cdots, s\} \\ g_{s+1}(x) &:= x^T \tilde{A}_1 x + \tilde{a}_{s+1}^T x \\ g_{s+2}(x) &:= x^T \tilde{A}_2 x + \tilde{a}_{s+2}^T x + \tilde{b}_{s+2} \\ g_{s+3}(x) &:= x^T \tilde{A}_3 x + \tilde{a}_{s+3}^T x + \tilde{b}_{s+3}. \end{aligned}$$

We easily verify that $P_I^\epsilon$ is a double quadratic problem. Therefore we consider

$\mathbf{P_I^\epsilon}(t)$ :

$$\min\{f(x, x_{n+1}, t) \mid g_j(x, x_{n+1}, t) \le 0, j \in J = \{1, \cdots, s+3\},$$

where

$$\begin{aligned} f(x, x_{n+1}, t) &:= \|(x, x_{n+1}) - (x^o, {x_{n+1}}^o)\|^2 \\ g_j(x, x_{n+1}, t) &:= g_j(x, x_{n+1}) \quad i \in \{1, \cdots, s\} \\ g_{s+1}(x, x_{n+1}, t) &:= g_{s+1}(x, x_{n+1}) \\ g_{s+2}(x, x_{n+1}, t) &:= g_{s+2}(x, x_{n+1}) \\ g_{s+3}(x, x_{n+1}, t) &:= g_{s+3}(x, x_{n+1}) + (t-1) g_{s+3}(x^o, x_{n+1}^o). \end{aligned}$$

We note that $(x^o, x_{n+1}^o)$ has to be chosen as in (C2).

**Remark 10.** *We see that $\hat{x}$ solves $\mathbf{P_I^\epsilon}$ if and only if $M_I^\epsilon = \emptyset$, for $\epsilon > 0$ sufficiently small. Here, we do not discuss the question: How can it be checked whether $M_I^\epsilon$ is empty or not? This is why we assume*

**(C8)** $$M_I^\epsilon \neq \emptyset.$$

Now we have to show whether the choice of the starting point is possible. According to Section 1, we define

$$\begin{aligned} &\mathcal{P}^\epsilon := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid g_j(x, x_{n+1}) \le 0, (j = 1, \cdots, s)\}, \\ &G_i^\epsilon := \{x \in \mathbb{R}^n \mid g_{s+i}(x, x_{n+1}) \le 0\}, \quad i = 1, \cdots, 3, \\ &H_1^\epsilon = \{x \in \mathbb{R}^{n+1} \mid D_x g_{s+3}(x) = 0\}. \end{aligned}$$

**Theorem 8 (Choice of a starting point).** *Let $\epsilon > 0$ be sufficiently small, and let $\hat{z} = (\hat{x}, \hat{x}_{n+1})$ be the solution of $IQOP_c$ and cl $int M_I^\epsilon = M_I^\epsilon$. Then we have*

$$int\mathcal{P}^\epsilon \cap int G_1^\epsilon \cap int G_2^\epsilon \setminus G_3^\epsilon \neq \emptyset.$$

*Proof.* Let $z := (x, x_{n+1})$ and suppose that

$$int\mathcal{P}^\epsilon \cap intG_1^\epsilon \cap intG_2^\epsilon \setminus G_3^\epsilon = \emptyset.$$

Then we have

$$(35) \qquad \forall z \in \mathbb{R}^{n+1} \quad z \notin int\mathcal{P}^\epsilon \cap intG_1^\epsilon \cap intG_2^\epsilon \text{ or } z \in G_3^\epsilon.$$

We distinguish 2 cases:

Case 1: $\hat{z} \in intP^\epsilon \cap intG_1^\epsilon \cap intG_2^\epsilon$
Then, we have

$$(36) \qquad \hat{z} \notin G_3^\epsilon,$$

since $g_{s+3}(\hat{z}) = \epsilon > 0$. From (35) we obtain:

$(a)$ $\hat{z} \notin int\mathcal{P}^\epsilon \cap intG_1^\epsilon \cap intG_2^\epsilon$ (contradiction to $\hat{z} \in M_I^c$).

$(b)$ $\hat{z} \in G_3^\epsilon$ (contradiction to (36)).

Case 2 : $\hat{z} \in \partial\mathcal{P}^\epsilon \cup \partial G_1 \cup \partial G_2$
Since *cl* $intM_I^c = M_I^c$, there exists a sequence $\{z_k\}$ with

$$\{z_k\} \subset intM_I^c$$
$$z_k \underset{\infty}{\longrightarrow} \hat{z}.$$

Hence, there is a $z_{k_o}$ such that $g_{s+3}(z_{k_o}) = f_I(z_{k_o}) - f_I(\hat{z}) + \epsilon > 0$. Thus $z_{k_o} \notin G_3^\epsilon$. Arguing as in case 1, the proof is complete. □

**Theorem 9.** *It holds*

$(a)$ $H_1^\epsilon = \emptyset$.

$(b)$ *The choice of the starting point* $(x^o, x_{n+1}^o)$ *for* $P_I^\epsilon$ *is, by the assumptions made in Theorem 8, always possible.*

According to the results above, we propose the following algorithm, which solves $(IQOP_c)$.

**Algorithm** 2

***Step 0*** Transform $(QOP)$ into $(\mathbf{IQOP})_\mathbf{c}$;

***Step 1*** Choose an $\epsilon > 0$ sufficiently small.
Compute a $\hat{z} \in \psi_{stat}(IQOP_c)(\psi_{gc}(IQOP)$;
Set $k := 0$, $x_k^o := \hat{z}$.

***Step 2*** If $M_{I_k}^\epsilon = \emptyset$, stop ($x_k^o$ is the solution). Else, go to step 3.

***Step 3*** If $\hat{z}$ satisfies $(C2)$, set $x_k^o = \hat{z}$. Else, compute a $\tilde{z}$ close to $\hat{z}$ which satisfies $(C2)$ and set $x_k^o = \tilde{z}$. Go to step 4.

***Step 4*** Call the algorithm 1:
Compute a g.c. point $p$ for $P_{I_k}^\epsilon(x_k^o, \hat{z})$. If the algorithm 1 is successful, set $k = k + 1$; $\hat{z} := p$; $x_k^o := \hat{z}$ and go to step 2. Else, stop.

### 5.1.2 The concave quadratic optimization problem

In view of the realization of Step2 we have:

$\mathbf{P}_\mathbf{c}^\epsilon$ :
$$\min\{f(x) = \|x - x^o\|^2 \mid x \in M_c^\epsilon\},$$

where

$$\begin{aligned} M_c^\epsilon &:= \{x \in \mathbb{R}^n \mid g_j(x) \le 0, j \in J\} \\ J &:= \{1, \cdots, m, m+1, \cdots, m+n+1\} \\ g_j(x) &:= a_j^T x + b_j, \qquad j \in \{1, \cdots, m\} \\ g_{m+i}(x) &:= -x_i, \qquad i \in \{1, \cdots, n\} \\ g_{s+1}(x) &:= f_c(x) - f_c(\hat{x}) + \epsilon, \qquad s := m+n, \end{aligned}$$

and $f_c(x) = x^T Q x + c^T x$. $P_c^\epsilon$ is a double quadratic problem.

For the choice of the starting point we assume:

$$cl\ int(\mathcal{P} \setminus G_{s+1}) = \mathcal{P} \setminus G_{s+1} \text{ and } H_2 \not\supseteq int(\mathcal{P} \setminus G_{s+1}).$$

Then we have the same results as in Theorem 8.

### 5.1.3 Illustrative examples

We consider Example 5.1 and try to find an approximate solution using the algorithm 2. After transformation we have

$\mathbf{IQOP_c}$ :
$$\min\{x_1^2 - x_3 \mid x \in \mathbb{R}^3, \quad g_j(x_1, x_2, x_3) \le 0, \quad j = 1, \cdots, 6\}$$

where

$$\begin{array}{lll} g_1(x_1, x_2, x_3) := x_1 - 10 & g_2(x_1, x_2, x_3) := -x_1 & \\ g_3(x_1, x_2, x_3) := -x_2 & g_4(x_1, x_2, x_3) := x_2 - 10.0 & \\ g_5(x_1, x_2, x_3) := x_3 - x_2^2 & g_6(x_1, x_2, x_3) := x_3^2 - 10000 & (compactification) \end{array}$$

Step 1: Take the stationary point $\hat{z} = (10.0, 10.0, 100)$.
Step 2: Find a g.c. point of

$\mathbf{P}_\mathbf{I}^\epsilon$ :
$$\min\{(x_1 - x_1^o)^2 + (x_2 - x_2^o)^2 + (x_3 - x_3^o)^2 \mid g_j(x_1, x_2, x_3) \le 0, \quad j = 1, \cdots, 7\},$$

where

$$g_7(x_1, x_2, x_3) := x_1^2 - x_3 - (\hat{z}_1 - \hat{z}_3) + \epsilon.$$

In the Figures 16, 17, 18 and 19 we have sketched and presented solutions of $P_{I_k}^\epsilon$, considering the iterations $k = 1, 7, 10$ (see also Table 1). The algorithm stops at the iteration 11 and more precisely, at a point of Type 5, where, locally the set $\sum_{gc}$ becomes empty (see the Fig. 19).

Iteration points

| $k$ | $\epsilon$ | $\hat{z}$ | $x^o$ | Solution of $P_{I_k}^{\epsilon}$ |
|---|---|---|---|---|
| 1 | 0.08 | $(10.0, 10.0, 100.0)$ | $(9.99, 9.99, 50.0)$ | $(7.08, 9.99, 50.24)$ |
| 2 | 0.08 | $(7.08, 9.99, 50.2)$ | $(7.08, 9.99, 50.24)$ | $(7.07, 9.99, 50.2)$ |
| 3 | 0.08 | $(7.07, 9.99, 50.2)$ | $(7.07, 9.99, 50.24)$ | $(7.06, 9.99, 50.2)$ |
| 4 | 0.5 | $(7.06, 9.99, 50.2)$ | $(7.06, 9.99, 50.24)$ | $(7.02, 9.99, 50.2)$ |
| 5 | 0.8 | $(7.02, 9.99, 50.2)$ | $(7.02, 9.99, 50.26)$ | $(6.96, 9.99, 50.2)$ |
| 6 | **10.0** | $(6.96, 9.99, 50.2)$ | $(6.96, 9.99, 50.26)$ | $(6.19, 9.99, 50.26)$ |
| 7 | **50.0** | $(6.19, 9.99, 50.26)$ | $(6.19, 9.99, 50.26)$ | $(0.25, 9.99, 61.97)$ |
| 8 | 0.8 | $(0.25, 9.99, 61.97)$ | $(0.25, 9.99, 61.97)$ | $(0.10, 9.99, 62.72)$ |
| 9 | 0.8 | $(0.10, 9.99, 62.72)$ | $(0.10, 9.99, 62.72)$ | $(0.04, 9.99, 63.51)$ |
| 10 | **20.0** | $(0.04, 9.99, 63.51)$ | $(0.04, 9.99, 63.51)$ | $(0.0001, 9.99, 83.51)$ |
| 11 | 0.8 | $(0.0001, 9.99, 83.51)$ | $(0.0001, 9.99, 83.51)$ | $(\mathbf{0.0}, \mathbf{10.0}, \mathbf{100})$ |

**Remark 11.** *The considered Example* 5.1*, computed with PAFO [11], shows that, although we can reach* $t = 1$ *for* $k \leq 20$ *and compute the solution, we are not able to check whether the computed solution is global, since we cannot decide whether* $M_I^{\epsilon} = \emptyset$*. At the iteration* $k = 10$ *we cannot reach* $t = 1$*, and PAFO stops at a point of Type* 5*, which is the solution. At this point* $M_I^{\epsilon}$ *becomes locally empty.*

## 5.2 Multiobjective optimization

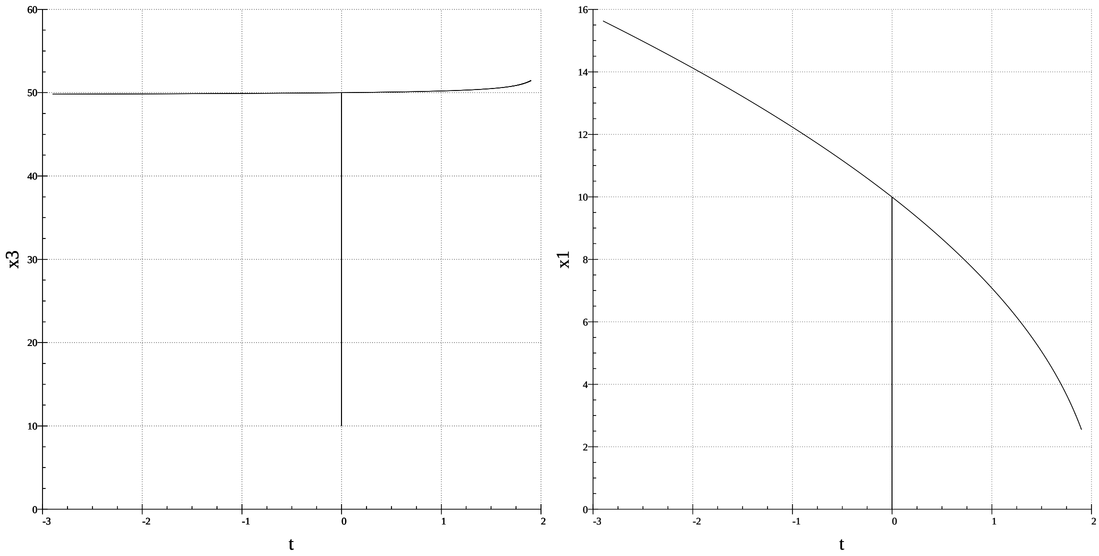
$$(MOP) \qquad \min\{f(x) \mid x \in \mathcal{P}\}, \quad f = (f_1, \cdots, f_l)^T,$$

where

$$\begin{aligned} f_1(x) &= x^T Q x + c_1^T x \\ f_j(x) &= c_j^T x + d_j, \quad j = 2, \cdots, l \\ \mathcal{P} &:= \{x \in \mathbb{R}^n \mid a_j^T x + b_j \leq 0, \quad (j = 1, \cdots, s)\} \end{aligned}$$

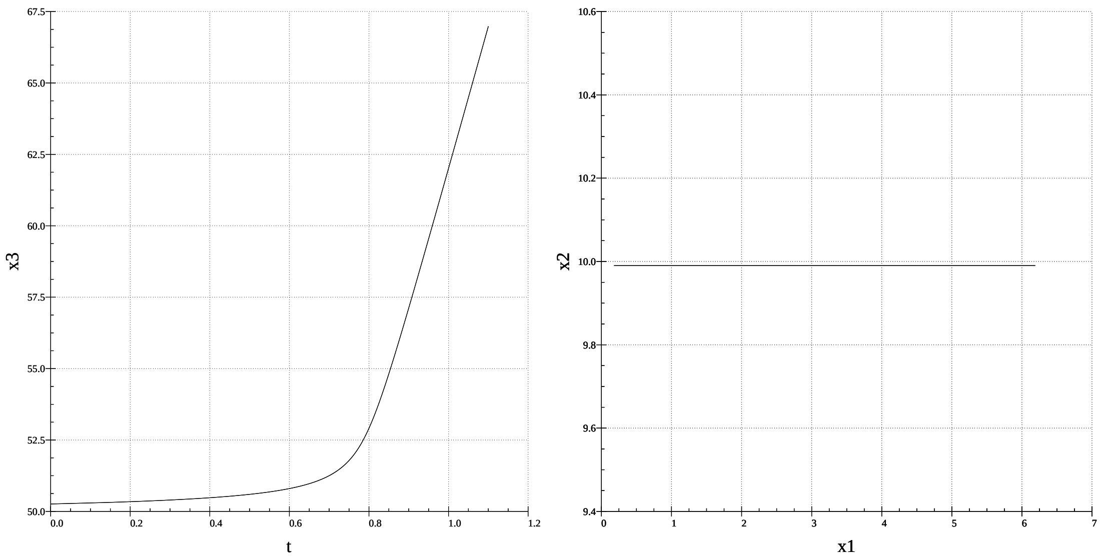
$\mathcal{P}$ is a convex polyhedron and the matrix $Q$ is negative semi-definite.
The main information consists in estimating the objective value at the point $x^i$ in comparison with $\underline{f_k} = inf\{f_k(x) | x \in \mathcal{P}\}$.

**Telescreen:**



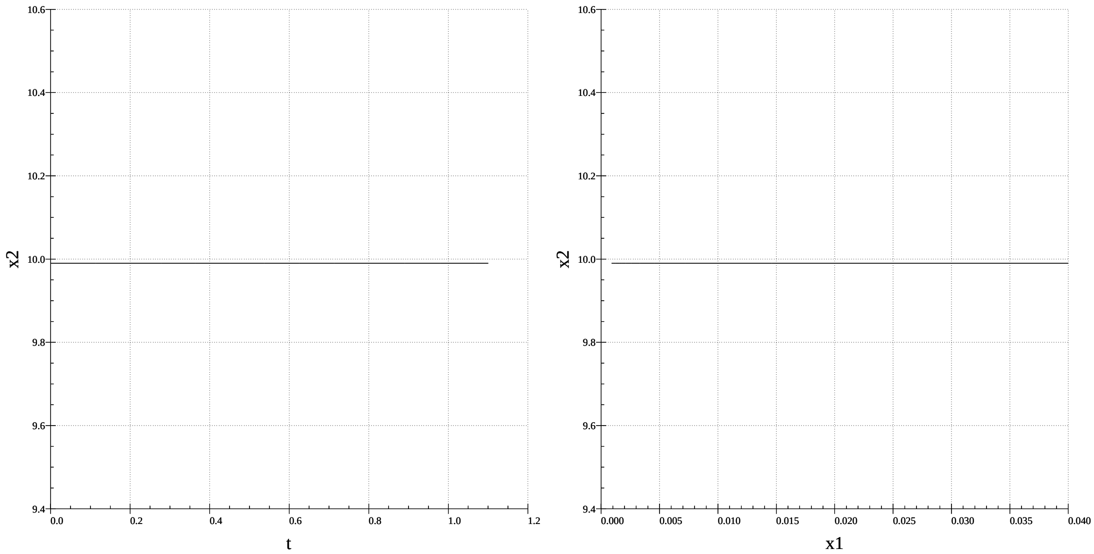

k=1; z=(10,10,100); x=(9.99,9.99,50); s=(7.08,9.99,50.2) k=1; x=(9.99,9.99,50); s=(7.08,9.99,50.2)

Figure 16: Iteration 1



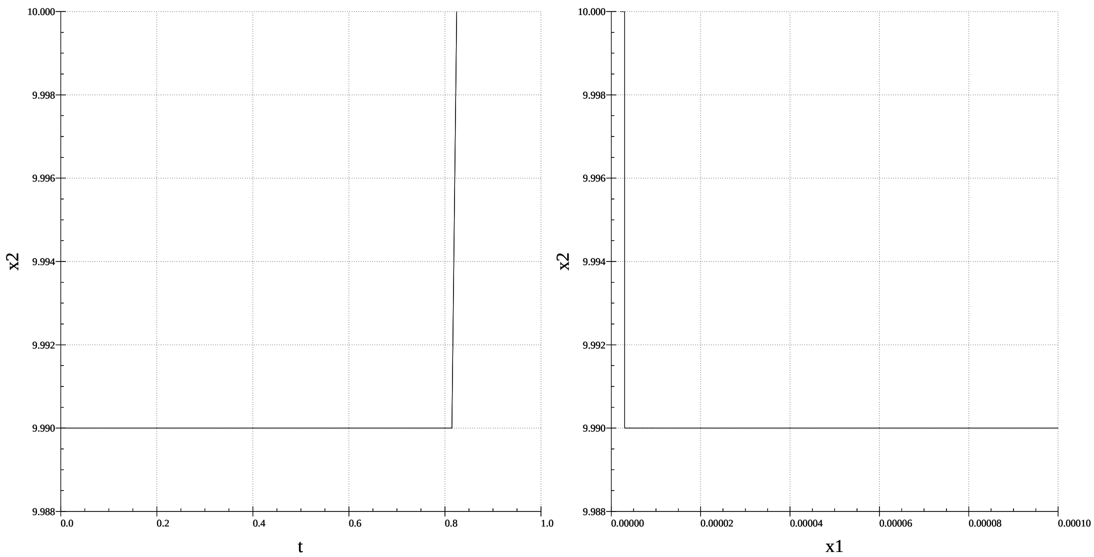

k=7; z=(6.19,9.99,50.26); x=z; s=(0.25,9.99,61.97) k=7; z=(6.19,9.99,50.26); x=z; s=(0.25,9.99,61.97)

Figure 17: Iteration 7




k=10; z=(0.04,9.99,63.51); x=z; s=(0.0001,9.99,83.51)　　x=z; s=(0.0001,9.99,83.51)

Figure 18: Iteration 10




global s=(0.0,10.00,100), type 5　　global s=(0.0,10.0,100), type 5

Figure 19: Iteration 11

$$
\begin{array}{ccc}
\underline{f}_1 & f_1(x^o) & \dfrac{f_1(x^o) - f_1}{|f_1|}.100 \\
\vdots & \vdots & \vdots \\
\underline{f}_k & f_k(x^o) & \dfrac{f_k(x^o) - f_k}{|f_k|}.100 \\
\vdots & \vdots & \vdots \\
\underline{f}_l & f_l(x^o) & \dfrac{f_l(x^o) - f_l}{|f_l|}.100
\end{array}
$$

We consider

$$M(\mu^1) := \{x \in \mathcal{P} \mid f_k(x) \leq \mu_l^1, k = 1, \cdots, l\}.$$

$\mu^1$ is the goal of the decision maker. In order to find the goal realizer $\hat{x} \in M(\mu^1)$, we propose to find a g.c. point of

$$\min\{||x - x^o||^2 \mid M(\mu^1) \cap E(p)\}, \quad \textit{where} \tag{37}$$

$$E(p) := \{x \in \mathbb{R}^n \mid ||x||^2 \leq p\}$$

, and $p \in \mathbb{R}$ is sufficiently large.

$$M(\mu^1) \cap E(p) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ccll} a_i^T x + b_i & \leq & 0 & i = 1, \cdots, s \\ f_j(x) - \mu_j^1 & \leq & 0 & j = 1, \cdots, l \\ ||x||^2 - p & \leq & 0 & \end{array} \right\},$$

Since (37) is a double quadratic problem, we can apply the results obtained above in order to find a goal realizer.

# References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